

Preface

These notes were assembled during the spring 2026 semester of the second-year PhD macroeconomics sequence at Penn State, taught by Maria-Jose Carreras-Valle (Part I) and Kai-Jie Wu (Part II). They aim to serve simultaneously as a compact reference for the technical machinery of modern macroeconomics—heterogeneous-agent equilibria, dynamic programming, business-cycle accounting, the empirics of consumption—and as a self-contained narrative of how the field’s central questions evolve from one chapter to the next.

Audience and Prerequisites

The intended reader is a first- or second-year graduate student who has had a careful undergraduate or master’s-level treatment of microeconomic theory (consumer choice, general equilibrium, basic dynamic programming) and the standard probability and real-analysis tools that come with that. No prior macroeconomics is strictly required, but the pace of *Part I* assumes familiarity with the Arrow–Debreu framework and the language of state-contingent claims.

Structure of the Book

The book is divided into two parts, reflecting the two-instructor structure of the course.

Part I: Heterogeneous Agents in Complete and Incomplete Markets (Chapters 1–3, by Maria-Jose Carreras-Valle) develops a unified framework for studying risk sharing across heterogeneous agents. Chapter 1 establishes the complete-markets benchmark—Arrow–Debreu trading, sequential trading, the recursive social planner—against which the rest of the book pushes. Chapter 2 introduces *exogenous* market incompleteness through Huggett, Aiyagari, and Krusell–Smith. Chapter 3 turns to *endogenous* incompleteness arising from participation frictions: one-sided lack of commitment, the Bulow–Rogoff model, and two-sided lack of commitment. The three chapters share a methodological signature: equilibria are characterized by the cross-sectional distribution of state variables, and the natural recursive formulation uses promised utility (or its analogue) as the state.

Part II: Growth, Business Cycles, and Quantitative Macroeconomics (Chapters 4–11, by Kai-Jie Wu) takes the dynamic-equilibrium machinery and applies it to canonical macroeconomic questions. Chapter 4 develops growth and development accounting as the empirical hook. Chapters 5–7 build the Solow and neoclassical growth models and confront them with cross-country convergence data. Chapter 8 extends to Real Business Cycles, and Chapter 9 inverts the RBC model to perform Business Cycle Accounting. Chapter 10

treats consumption and saving theory—the Permanent Income Hypothesis, Hall’s Random Walk Hypothesis, and the empirical literature documenting excess sensitivity. Chapter 11 closes with the computation of the Aiyagari heterogeneous-agent model, which serves as the bridge into the modern HANK literature.

Pedagogical Conventions

Several typographic conventions recur throughout the text.

- **Definitions** appear in green-shaded boxes. **Theorems, Propositions, Lemmas, Corollaries**, and **Claims** appear in cyan-shaded boxes; their proofs follow inline (or in a dedicated grey-bordered block, when emphasized).
- **Remarks** come in two flavors. The shorter *inline* remarks (`\rmk`) flag a brief point in the surrounding narrative; the boxed *block* remarks (`\rmkb`) develop a substantial side topic, often spanning several paragraphs and including subsidiary figures or tables.
- **Algorithms** (e.g. Value Function Iteration, Aiyagari’s outer loop) appear in violet-shaded boxes, listing the steps in order with implementation notes.
- **Examples** appear in their own environment with the worked solution clearly demarcated.
- **Facts** report empirical regularities in their own boxes, typically appearing in chapters that confront theory with data.

Each chapter opens with a brief *Notation in This Chapter* table listing chapter-specific symbols. The book-wide *Notation* section (immediately following this preface) collects symbols common to multiple chapters.

Reading Paths

Readers do not have to proceed linearly.

- *Heterogeneous-agent macro focus*. Read Part I in full, then Chapter 11 (Aiyagari computation). Chapter 10’s PIH section provides useful background for the household problem in Aiyagari but is not strictly required.
- *Growth focus*. Read Chapters 4–7 as a self-contained block on growth theory and its cross-country evidence.
- *Business cycles focus*. Chapters 8–9 are the core; Chapter 10’s RWH section complements the empirical discussion.
- *Computational focus*. Chapter 6 (Section on VFI), Chapter 8 (RBC numerical solution), and Chapter 11 (Aiyagari) form a sequence of progressively harder computational exercises.

Acknowledgments

These notes would not exist without Maria-Jose Carreras-Valle and Kai-Jie Wu, whose lectures form the underlying material. Any errors are mine—both as the typesetter and as the student.

Rui Zhou, Spring 2026

Notation

The following symbols recur throughout the notes. Where a chapter departs from a convention listed here, a chapter-specific note is provided in its opening section. A few high-level conventions:

- **Lowercase vs. uppercase letters.** Lowercase letters (e.g. c, k, y) denote per-worker or per-capita quantities. Uppercase letters (e.g. C, K, Y) denote aggregates. The convention is occasionally relaxed in specific chapters; when it matters, the chapter's notation note flags the exception.
- **Time subscripts.** t indexes the period; T is the terminal period in finite-horizon problems and the simulation length in numerical sections.
- **States and histories.** $s_t \in S$ is the period- t exogenous state; $s^t = (s_0, s_1, \dots, s_t)$ is the history through date t .
- **Conditional expectation.** $\mathbb{E}_t[\cdot]$ denotes expectation conditional on the time- t information set.

Symbols used throughout the book.

| Symbol | Meaning |
|------------------------------------|---|
| <i>Preferences and discounting</i> | |
| $u(\cdot)$ | Period utility function; $u' > 0$, $u'' < 0$, satisfying Inada conditions where needed. |
| β | Time discount factor; $\beta \in (0, 1)$. |
| σ | Coefficient of relative risk aversion under CRRA utility; the inverse $1/\sigma$ is the intertemporal elasticity of substitution. |
| γ | Coefficient of <i>absolute</i> risk aversion under CARA utility (Ch. 2 only). |
| $\mathbb{E}_t[\cdot]$ | Expectation conditional on history s^t . |
| <i>Stochastic environment</i> | |
| s_t, s^t | Date- t state; history through t . |
| $\pi(s^t)$ | Unconditional probability of history s^t ; $\pi(s^\tau s^t)$ is conditional. |
| ε_t | Innovation / shock realization. |
| ρ | Persistence parameter of an AR(1) process; $\rho = \psi$ in Ch. 2's CARA example. |
| <i>Endowment and production</i> | |
| $y(s^t), Y_t$ | Stochastic endowment; aggregate output. |

(continued on next page)

| Symbol | Meaning |
|--|--|
| $F(K, L)$ | Aggregate production function, typically constant returns to scale. |
| $f(k)$ | Per-worker production function $f(k) = F(k, 1)$. |
| A, a_t | Total factor productivity (TFP); $a_t = \ln A_t$ for the log-linear AR(1) version. |
| α | Capital share in Cobb–Douglas production; output elasticity of capital. |
| δ | Depreciation rate of physical capital; $\delta \in (0, 1]$. |
| <i>Quantities</i> | |
| c, C | Consumption (per worker / aggregate). |
| k, K | Physical capital (per worker / aggregate). |
| L, l | Labor (aggregate / per worker). $L = 1$ in many setups. |
| I_t | Aggregate investment, $I_t = K_{t+1} - (1 - \delta)K_t$. |
| a, A | Asset / debt holdings (note: A is also used for TFP and natural debt limit; context disambiguates). |
| <i>Prices and returns</i> | |
| r | Real interest rate. Convention varies: in Ch. 1–3, 5–10, r is the net rate or rental rate of capital; in Ch. 11, $r = F_K(K, L)$ is the rental rate and the household’s gross return is $1 + r - \delta$. Each chapter’s notation note specifies the convention used. |
| R | Gross interest rate; typically $R = 1 + r$. |
| w | Real wage. |
| $q(s^t)$ | Date-0 Arrow–Debreu price of a state-contingent claim (Ch. 1). |
| $Q(s^t s)$ | One-period-ahead pricing kernel in sequential trading (Ch. 1, 2). |
| <i>Solution objects</i> | |
| V | Value function. |
| $g(\cdot)$ | Policy function. |
| Λ, λ | Cross-sectional distribution of agents (Ch. 2, 11). |
| <i>Lagrangian and shadow prices</i> | |
| \mathcal{L} | Lagrangian. |
| λ^i, μ^i | Pareto weight or Lagrange multiplier on a specific agent’s budget; context distinguishes from the distribution λ . |
| $\theta(s^t)$ | Multiplier on resource constraint (planner’s problem, Ch. 1). |
| <i>Empirical / decomposition objects</i> | |
| Var, Cov | Cross-sectional variance and covariance. |
| g_x | Average growth rate of variable x over a sample period (Ch. 4). |

A few overloaded symbols deserve attention. The Greek letter λ is used both for Pareto weights / Lagrange multipliers and for the cross-sectional distribution of agents—the role is always clear from context. The letter A is used for both the natural debt limit (Ch. 1) and TFP (Ch. 5 onward); these never appear together. The letter a is used for asset holdings throughout, and as log-TFP in Ch. 8; again no overlap.

Each chapter opens with a brief notation note flagging any chapter-specific symbols and confirming the local interpretation of r and a few other context-dependent objects.

Part I

Heterogeneous Agents in Complete and Incomplete Markets

Lectures by Maria-Jose Carreras-Valle

Part II

Growth, Business Cycles, and Quantitative Macroeconomics

Lectures by Kai-Jie Wu

Chapter 4

Growth and Development Accounting

Remark (Notation in This Chapter).

| Symbol | Meaning |
|----------------------------------|--|
| $Y_{it}, K_{it}, H_{it}, L_{it}$ | Output, physical capital, total human capital, raw labor force in country i at t |
| y, k, h | Per-worker counterparts |
| α | Capital share / output elasticity in Cobb–Douglas $y = Ak^\alpha h^{1-\alpha}$ |
| $j \in \{1, \dots, J\}$ | Schooling group index |
| s_j | Years of schooling for group j |
| l_j | Population share of schooling group j |
| h_j | Productivity weight for schooling group j (relative to $h_1 \equiv 1$) |
| w_j | Wage rate for schooling group j |
| m | Mincer return to one additional year of schooling |
| \hat{h}_{it} | Empirical estimator for human capital using Mincer wage premium |
| g_x | Average growth rate of variable x over a sample period |
| s_k, s_h, s_A | Decomposition shares of k, h, TFP in income variation / growth |
| X_i | Composite of observable inputs: $X_i \equiv \alpha \ln k_i + (1 - \alpha) \ln h_i$ |

This chapter transitions from theoretical dynamics to empirical measurement through an exercise known as **Accounting** (i.e., variance and growth decomposition). The fundamental goal of this chapter is to open the “black box” of aggregate output and attribute it to observable factors versus unobservable productivity. We ask two distinct quantitative questions:

1. **Development Accounting (Cross-Sectional):** Why are some countries vastly richer than others at a given point in time? How much of this income gap is due to differences in physical and human capital, versus differences in Total Factor Productivity (TFP)?
2. **Growth Accounting (Time-Series):** Why does a specific country grow over time?

How much of its historical growth rate can be attributed to the accumulation of capital, versus the growth of TFP?

Typically, empirical evidence suggests that physical capital accumulation accounts for 30–40% of growth, human capital for about 10% (though debatable), and the remaining roughly 50% is attributed to improving productivity.

4.1 The Production Framework: From General to Specific

Assume we observe a panel dataset of economies $i \in I$ over time periods $t \in T$:

$$\{Y_{it}, K_{it}, H_{it}, L_{it}\}_{i \in I, t \in T}$$

where Y is total output, K is physical capital, H is total human capital (efficiency units of labor, defined later), and L is the raw labor force.

The aggregate production function is given by:

$$Y_{it} = A_{it}F(K_{it}, H_{it})$$

where $F(\cdot)$ is assumed to exhibit *constant returns to scale (CRS)*.

Here, it is crucial to distinguish between the physical properties of the inputs:

- **Rivalrous Inputs (K_{it}, H_{it}):** A good is defined as *strictly rivalrous* if its use or consumption by one economic agent fundamentally precludes its simultaneous use by another. Physical capital and human capital are structurally rivalrous because they are strictly bound by physical space and time constraints. For instance, a single machine (K) cannot be simultaneously operated by multiple workers without severe congestion. Similarly, human capital (H) is embodied within the worker; an hour of a worker's skilled labor deployed in one factory physically cannot be concurrently deployed in another.
- **Non-Rivalrous Productivity (A_{it}):** The Total Factor Productivity (TFP) term A_{it} captures non-rivalrous goods—such as knowledge, blueprints, software, and institutional quality. An idea can be shared simultaneously across all workers and machines without being depleted.

Because $F(\cdot)$ is CRS, we can divide both sides of the equation by the raw labor force L_{it} to express the economy in per-worker terms. Crucially, because A_{it} is non-rivalrous, this transformation does not dilute it; every worker can fully access exactly the same level of

technology¹.

$$y_{it} = A_{it}F\left(\frac{K_{it}}{L_{it}}, \frac{H_{it}}{L_{it}}\right) \equiv A_{it}f(k_{it}, h_{it})$$

where y_{it} is output per worker, k_{it} is physical capital per worker, and h_{it} is human capital per worker.

While the general function $f(\cdot)$ establishes the abstract theoretical mechanics, performing an actual quantitative accounting exercise requires concrete numbers. Specifically, we need to know *by exactly what percentage* output changes when an input changes—this is the output elasticity. To measure and decompose variances in the real-world data, we must parameterize these output elasticities into specific, measurable constants. The universal standard in the macro literature is to adopt the Cobb-Douglas functional form, which conveniently assumes a constant capital share (elasticity) $\alpha \in (0, 1)$:

$$y_{it} = A_{it}k_{it}^{\alpha}h_{it}^{1-\alpha}$$

Taking the natural logarithm of both sides, we obtain the fundamental log-linearized accounting identity:

$$\ln y_{it} = \ln A_{it} + \alpha \ln k_{it} + (1 - \alpha) \ln h_{it}$$

This linear additive form is the starting point for all variance and growth decompositions. The fundamental question of macro accounting is: How much of the variation in $\ln y_{it}$ is driven by the observable inputs (k_{it}, h_{it}) versus the unobservable TFP residual (A_{it}) ? Before answering it, the next section addresses two prerequisite measurement problems: how to pin down α , and how to measure human capital h_{it} .

4.2 Measurement: α and Human Capital

4.2.1 How to Pin Down α ?

To conduct the accounting exercise, we need the value of α . A naive approach is to simply run an OLS regression of $\ln y_i$ on $\ln k_i$ and $\ln h_i$ to extract the coefficient. However, this suffers from severe *endogeneity (omitted variable bias)*. The error term $\ln A_i$ is likely to be positively correlated with the regressors (e.g., higher tech countries attract more capital k_i), leading OLS to overestimate α .

There are typically two ways to cope with this endogeneity problem:

1. **Econometrics:** Find an IV that is orthogonal to $\ln A_i$ and run a 2SLS regression. However, this is often difficult in practice, and the results can be sensitive to the choice of instrument.

¹Algebraically, because $F(\cdot)$ is CRS, dividing both sides by L_{it} allows us to pass $1/L_{it}$ inside the function F , yielding $y_{it} = A_{it}f(k_{it}, h_{it})$. However, this mathematical legality must be strictly backed by the underlying physical properties in economics.

Imagine if A_{it} were not abstract technology, but rather a pile of “coal” (a rivalrous good). The original production function would have to be written as $Y_{it} = F(K_{it}, H_{it}, A_{it})$. When converting this economy to per-worker terms, the coal allocated to each worker would inevitably be diluted to $a_{it} = A_{it}/L_{it}$.

The only reason we are allowed to place A_{it} *outside* the function F as a global multiplier—and completely exempt it from being divided by L_{it} during the per-worker transformation—is because we have predefined technology as strictly non-rivalrous. Regardless of how much the population L_{it} expands, the “calculus formula” (A_{it}) available to each worker remains the complete A_{it} , rather than a diluted A_{it}/L_{it} .

2. **Economic Theory (the standard approach):** Model the bias away using microeconomics.

Assuming perfectly competitive markets², the representative firm's profit maximization problem is:

$$\max_{k_i, h_i} A_i k_i^\alpha h_i^{1-\alpha} - r k_i - w h_i$$

Taking the FOC with respect to k_i gives the marginal product of capital:

$$\alpha A_i k_i^{\alpha-1} h_i^{1-\alpha} = r \implies \alpha \frac{y_i}{k_i} = r$$

Rearranging this yields:

$$\alpha = \frac{r k_i}{y_i}$$

This shows that α is exactly equal to the *capital income share* in the national accounts. We can simply look up this share in the data without running any biased regressions.

4.2.2 How to Measure Human Capital h_{it} ?

Following the seminal work of ?, we construct aggregate human capital by linking macroeconomic aggregates to microeconomic wage data.

Suppose a country has heterogeneous workers divided into schooling groups $j \in \{1, 2, \dots, J\}$ (e.g., No schooling, Elementary, High school, College). Each group has s_j years of schooling, and makes up a share l_j of the population. Let group 1 be the baseline uneducated workers ($h_1 = 1$).

Aggregate human capital H is the sum of efficiency units of labor:

$$H = \sum_{j=1}^J h_j L_j$$

This equation converts a heterogeneous workforce into a single, homogeneous macroeconomic measure called “efficiency units of labor.” Specifically, L_j is the absolute number of workers in schooling group j , and h_j captures their specific human capital level (or relative productivity weight). By normalizing $h_1 = 1$ for uneducated workers, H essentially measures the total labor input expressed in terms of “uneducated-worker equivalents.”

The firm's problem is to choose capital and the amount of labor from each group to maximize profit:

$$\max_{K, \{L_j\}} AK^\alpha \left(\sum_{j=1}^J h_j L_j \right)^{1-\alpha} - rK - \sum_{j=1}^J w_j L_j$$

The FOC with respect to labor type j equates the marginal product of labor to its wage:

$$w_j = (1 - \alpha) AK^\alpha H^{-\alpha} \cdot h_j = \left((1 - \alpha) \frac{Y}{H} \right) h_j$$

²In standard microeconomic theory, the assumption of a perfectly competitive market implies that the representative firm is a strict *price taker*. Consequently, the factor prices—the rental rate of capital (r) and the wage rate (w)—are exogenous and fixed from the firm's perspective. This allows us to equate the marginal product of capital directly to r without modeling the firm's impact on market prices.

By dividing the FOC of group j by the FOC of the uneducated group 1, we obtain the relative wage ratio:

$$\frac{w_j}{w_1} = \frac{h_j}{h_1} = h_j$$

This shows that *relative wages perfectly reflect relative human capital (productivity)*. Thus, the aggregate and per-capita human capital measures H and h can be expressed as³:

$$\hat{H} = \sum_{j=1}^J \left(\frac{w_j}{w_1} \right) L_j$$

$$\hat{h} = \sum_{j=1}^J \left(\frac{w_j}{w_1} \right) l_j$$

4.2.3 The Mincer Equation

How do we find the wage ratio w_j/w_1 ? We rely on the ? wage regression from labor economics:

$$\ln w_i = m s_i + \beta X_i + \varepsilon_i$$

Here, m is the *Mincer return*, capturing the percentage wage gain from one additional year of schooling (s_i). X_i is a vector of controls. This equation implies:

$$\ln w_j - \ln w_1 = m(s_j - s_1) \implies \frac{w_j}{w_1} = e^{m(s_j - s_1)}$$

Substituting the Mincerian wage premium back into our human capital index, we get the final measurement formula:

$$\hat{h}_{it} = \sum_{j=1}^J e^{m_{it}(s_{jit} - s_{1it})} l_{jit}$$

where

- i : denotes the specific country (the cross-sectional dimension).
- t : denotes the specific year (the time-series dimension).
- j : denotes the specific schooling group within that country and year.

To compute this across countries and time, we combine two datasets:

- $\{l_{jit}, s_{jit}\}$: Population shares and years of schooling from ?.
- $\{m_{it}\}$: Estimates of Mincerian returns to schooling from ?.

³The “hat” ($\hat{\cdot}$) explicitly denotes that this is an *empirical estimate* or proxy. The true human capital level h is intrinsically unobservable. However, by leveraging the microeconomic equilibrium condition that wage ratios equal productivity ratios, we can construct the estimator \hat{h} using purely observable market data (relative wages and population shares).

4.3 Development Accounting

Fix a specific year t , yielding a cross-sectional dataset $\{y_i, k_i, h_i\}_{i \in I}$. To simplify notation, we aggregate the observable inputs into a single “factor composite” X_i :

$$X_i \equiv \alpha \ln k_i + (1 - \alpha) \ln h_i$$

Thus, the income variation is perfectly decomposed into observable factors and unobservable TFP:

$$\ln y_i = X_i + \ln A_i$$

- **Method 1: The Simple Variance Ratio**

A naive approach measures the contributions as:

$$s_k^{(1)} = \frac{\text{Var}[\alpha \ln k_i]}{\text{Var}[\ln y_i]}, \quad s_h^{(1)} = \frac{\text{Var}[(1 - \alpha) \ln h_i]}{\text{Var}[\ln y_i]}$$

Limitations: This method has two major statistical flaws. First, variances cannot capture the direction of the relationship. Second, because $\text{Var}[\ln y_i] = \text{Var}[X_i] + \text{Var}[\ln A_i] + 2 \text{Cov}[X_i, \ln A_i]$, the shares will not sum to 100% unless the covariance between observable factors and TFP is exactly zero (which is strongly rejected in the data).

- **Method 2: Covariance-Based Decomposition**

To address these limitations, the modern literature (e.g., Klenow and Rodríguez-Clare, 1997) standardly adopts a covariance-based approach:

$$s_k = \frac{\text{Cov}[\alpha \ln k_i, \ln y_i]}{\text{Var}[\ln y_i]}, \quad s_h = \frac{\text{Cov}[(1 - \alpha) \ln h_i, \ln y_i]}{\text{Var}[\ln y_i]}$$

Intuition: Notice that s_k is exactly the slope coefficient β_k from a bivariate OLS regression of $\alpha \ln k_i$ on $\ln y_i$. Furthermore, because covariance is a linear operator, this method splits the joint covariance term linearly and guarantees exact additivity: $s_k + s_h + s_A = 1$ (if we in addition define $s_A = \frac{\text{Cov}[\ln A_i, \ln y_i]}{\text{Var}[\ln y_i]}$).⁴

4.4 Growth Accounting

While development accounting focuses on the cross-country differences in income *levels* at a given point in time, *growth accounting* aims to decompose the changes in income *over time*.

Suppose we have successfully pinned down the capital share α and observed the panel dataset:

$$\{y_{it}, k_{it}, h_{it}\}_{i \in I, t \in T}$$

⁴By the definition of variance, we know that $\text{Var}[\ln y_i] = \text{Cov}[\ln y_i, \ln y_i]$. Substituting the income identity $\ln y_i = X_i + \ln A_i$ into the first argument of the covariance yields:

$$\text{Var}[\ln y_i] = \text{Cov}[X_i + \ln A_i, \ln y_i] = \text{Cov}[X_i, \ln y_i] + \text{Cov}[\ln A_i, \ln y_i]$$

Dividing both sides by $\text{Var}[\ln y_i]$, we immediately obtain $s_X + s_A = 1$ (where $s_X = s_k + s_h$).

4.4.1 Time-Series Decomposition for a Single Economy

Pick a specific country i and observe its trajectory over time. We start with the log-linearized production function:

$$\ln y_t = \ln A_t + \alpha \ln k_t + (1 - \alpha) \ln h_t$$

Differentiating across time, we can express this relationship in terms of growth rates:

$$g_y = g_A + \alpha g_k + (1 - \alpha) g_h$$

where g_x denotes the average growth rate of variable x . This additive equation allows us to directly compute the contribution share of each factor to the overall economic growth of that country:

$$s_k = \frac{\alpha g_k}{g_y}, \quad s_h = \frac{(1 - \alpha) g_h}{g_y}, \quad s_A = \frac{g_A}{g_y}$$

By construction, these shares sum exactly to 1 (i.e., $s_k + s_h + s_A = 1$).

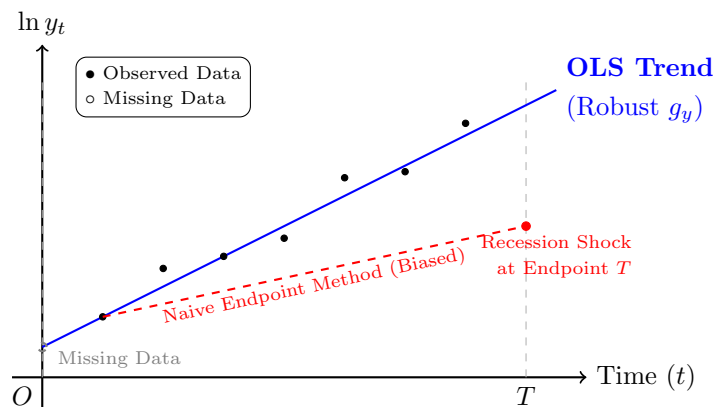
Remark (How to Compute the Average Growth from the Data?).

Take the computation of g_y as an example. A naive method is to use the endpoints: $g_y = (\ln y_T - \ln y_0)/T$. However, if the dataset is unbalanced (e.g., missing data for the exact years 0 or T), this method completely fails. Moreover, endpoint calculations are highly sensitive to business cycle fluctuations (e.g., year T happens to be a recession).

A common practice is to regress the logarithm of the variable on time t :

$$\ln y_t = \beta_0 + g_y \cdot t + \varepsilon_t$$

The OLS slope coefficient yields a robust estimate of the average growth rate g_y . Because OLS utilizes *all* available interior data points to fit the trend line, it naturally bypasses the issue of missing endpoints and smooths out short-term economic shocks.



4.4.2 Cross-Sectional Decomposition of Growth Rates

The second major question in growth accounting is: **Why do some countries grow faster than others?**

To answer this, we shift our focus to the cross-sectional dispersion of growth rates.

Suppose we have computed the average growth rates for all countries, yielding the following cross-sectional dataset:

$$\{g_{y_i}, g_{k_i}, g_{h_i}\}_{i \in I}$$

Since $g_{y_i} = g_{A_i} + \alpha g_{k_i} + (1 - \alpha)g_{h_i}$, we can decompose the *variance* of output growth rates ($\text{Var}[g_{y_i}]$) across countries. The methodology mirrors development accounting.

- **Method 1: The Simple Variance Ratio**

We could naively define the contributions as:

$$s_k^{(1)} = \frac{\text{Var}[\alpha g_{k_i}]}{\text{Var}[g_{y_i}]}, \quad s_h^{(1)} = \frac{\text{Var}[(1 - \alpha)g_{h_i}]}{\text{Var}[g_{y_i}]}$$

As discussed in Development Accounting, this method is statistically flawed. Because input growth rates and TFP growth are often correlated (e.g., fast technological adoption induces rapid capital accumulation), the covariance terms are non-zero. Consequently, the shares under Method 1 will not sum up to 100%.

- **Method 2: Covariance-Based Decomposition**

To ensure exact additivity, the standard literature employs the covariance-based approach:

$$s_k = \frac{\text{Cov}[\alpha g_{k_i}, g_{y_i}]}{\text{Var}[g_{y_i}]}, \quad s_h = \frac{\text{Cov}[(1 - \alpha)g_{h_i}, g_{y_i}]}{\text{Var}[g_{y_i}]}, \quad s_A = \frac{\text{Cov}[g_{A_i}, g_{y_i}]}{\text{Var}[g_{y_i}]}$$

Intuition: Once again, s_k can be visualized as the slope coefficient from an OLS regression of the capital contribution (αg_{k_i}) on the overall growth rate (g_{y_i}). This method allocates the covariance between factor accumulation and technological progress, guaranteeing that $s_k + s_h + s_A = 1$.

Remark (Chapter Summary).

- **Cobb–Douglas as the universal accounting framework.** Constant capital share α is read directly off national-accounts data, sidestepping the OLS endogeneity that would arise from estimating α via a regression of $\ln y$ on $\ln k, \ln h$.
- **Mincer wage equation operationalizes human capital.** Relative wages reflect relative productivity: $w_j/w_1 = e^{m(s_j - s_1)}$. The Mincer return m comes from labor-economics regressions; population shares come from Barro–Lee.
- **Method 2 (covariance-based) is preferred over the variance ratio.** It guarantees additivity ($s_k + s_h + s_A = 1$) by allocating the cross-term symmetrically, and admits the OLS-slope interpretation.
- **The empirical bottom line.** Across many decompositions: physical capital accounts for ~ 30 – 40% of growth, human capital for $\sim 10\%$, and TFP for the remaining $\sim 50\%$. This motivates the Solow framework that follows: TFP, not factor accumulation, is what we ultimately need to explain.
- **Same machinery serves cross-section and time series.** Development accounting decomposes income variance across countries; growth accounting decomposes growth-rate variance over time or across countries. The mathematics is parallel.

Part III

Problem Sets and Solutions

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