Please don't remove staple, thank you. *Student name:* ....

Econ 501 Final — December 12 2022

You are allowed one piece of letter size paper with writing on one side only. Show your work throughout.

1. Suppose that $\{\boldsymbol{x}_i\}$ are i.i.d. draws from a distribution with density function

$$f(x) = 2\max(x,0)\exp(-x^2/\theta_0)/\theta_0,$$

for some $\theta_0 > 0$.

(a) Derive the asymptotic distribution of the maximum likelihood estimator of θ_0. Express the asymptotic variance in terms of θ_0 only.
Answer: $N(0, \theta_0^2)$

(b) Propose a test for the hypothesis $H_0 : \theta_0 = 1$ versus $H_1 : \theta_0 \neq 1$ at a 5% significance level. Please be complete in your answer, i.e. include a decision rule, a test statistic, and a critical value.
Answer: several possibilities, but an ordinary t test with $\boldsymbol{t} = n^{-1}\sum_{i=1}^n(\boldsymbol{x}_i^2 - 1)$ does fine. The critical value is then 1.96.

(c) Show that your test is consistent.
Answer: like in class

(d) Determine the local power of your test if the local alternative is $\theta_0 = 1 + \lambda/\sqrt{n}$.
Answer: like in class

2. Let

$$\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} \sim N\left(0, \begin{bmatrix}1 & \rho^\intercal\\ \rho & I\end{bmatrix}\right),$$

for some vector ρ whose elements are all between -1 and 1. Express $\mathbb{V}(\boldsymbol{y} \mid \boldsymbol{x} \leq x)$ in terms of ρ, x and the inverse Mills ratio $m(x) = \phi(x)/\Phi(x)$ where ϕ, Φ are the standard normal density function and standard normal distribution function, respectively. **[2]**
Answer: Note that $\mathbb{E}(\boldsymbol{x}|\boldsymbol{x} \leq x) = -m(x)$, $\mathbb{E}(\boldsymbol{x}^2|\boldsymbol{x} \leq x) = 1 - xm(x)$, $\mathbb{E}(\boldsymbol{y}|\boldsymbol{x} \leq x) = -xm(x)\rho$, $\mathbb{E}(\boldsymbol{y}\boldsymbol{y}^\intercal \mid \boldsymbol{x} \leq x) = I - xm(x)\rho\rho^\intercal$. Thus, the answer is $I - m(x)\{x + m(x)\}\rho\rho^\intercal$

3. Suppose that $2n(\boldsymbol{x}_n - 1) \xrightarrow{d} N(0,1)$ and $\boldsymbol{y}_n \xrightarrow{p} 2$. Derive the limit distribution of $\{n(\boldsymbol{x}_n^2 - 1)\}^{\boldsymbol{y}_n}$.

4. Recall that the Hoeffding inequality says that if $\{\boldsymbol{x}_i\}$ is independent (but not necessarily identically distributed) and for some numbers $\{a_i\}$, $\{b_i\}$, $\mathbb{P}(a_i \leq \boldsymbol{x}_i \leq b_i) = 1$ then

$$\forall z > 0 : \mathbb{P}\left(\left|\sum_{i=1}^n(\boldsymbol{x}_i - \mathbb{E}\boldsymbol{x}_i)\right| > z\right) \leq 2\exp\left(\frac{-2z^2}{\sum_{i=1}^n(b_i - a_i)^2}\right).$$

Suppose that the support of the i.i.d. $\{\boldsymbol{x}_i\}$ is $[-1/\sqrt{2}, 1/\sqrt{2}]$. Construct a test for the hypothesis $H_0 : \mu_0 = 0$ against the alternative hypothesis $H_1 : \mu_0 \neq 0$ where $\mu_0 = \mathbb{E}\boldsymbol{x}_1$ that

has rejection probability no greater than α under H_0. Use a decision rule of the form: 'reject H_0 if $|\bar{\boldsymbol{x}}| > C_\alpha$.'
Answer: Reject H_0 if $|\bar{\boldsymbol{x}}| > \sqrt{n^{-1}\log(2/\alpha)}$.

5. True or false? Please provide a justification if true and a counter example if false.

   (a) $\forall x, y : F_{\boldsymbol{x}|\boldsymbol{y}}(x|y)F_{\boldsymbol{y}}(y) = F_{\boldsymbol{xy}}(x,y)$, where the $F_{\boldsymbol{x}|\boldsymbol{y}}$ denotes a conditional distribution function, $F_{\boldsymbol{xy}}$ a joint distribution function and $F_{\boldsymbol{y}}$ a marginal distribution function.
   Answer: False generally: only true if there is independence. If $f(x,y) = 2\mathbb{1}(0 \le x \le y \le 1)$ then $F(x|y) = \mathbb{1}(x \le y)\min(x,y)/y$, $F(y) = y^2$, and $F(x,y) = (2xy - x^2)\mathbb{1}(x \le y)$ for $0 \le x, y \le 1$.

   (b)
   $$\forall u, v : \frac{\mathbb{E}\exp\{i(u\boldsymbol{x} + v\boldsymbol{y})\}}{\mathbb{E}\exp(iu\boldsymbol{x})\mathbb{E}\exp(iv\boldsymbol{y})} = 1 \Rightarrow \boldsymbol{x}, \boldsymbol{y} \text{ are independent.}$$

   Answer: true: if the joint characteristic function is the product of the marginal characteristic functions then there is independence

   (c) $\boldsymbol{x}$ is conditionally mean independent of $\boldsymbol{y} \Rightarrow \boldsymbol{y}$ is conditionally mean independent of $\boldsymbol{x}$.
   Answer: False. Take $\boldsymbol{y} = \boldsymbol{x}^2$ where $\boldsymbol{x}$ has an even distribution. Then $\mathbb{E}(\boldsymbol{x} \mid \boldsymbol{y} = y) = 0$ for all y, but $\mathbb{E}(\boldsymbol{y} \mid \boldsymbol{x} = x) = x^2$.

6. Suppose that $\theta_0 = \arg\min_{\theta\in\Theta}\Omega(\theta)$ (so θ_0 is unique) where Ω is continuous and Θ compact. Suppose further that $\sup_{\theta\in\Theta}|\hat{\boldsymbol{\Omega}}(\theta) - \Omega(\theta)| = o_p(1)$. Prove that $\hat{\boldsymbol{\theta}} = \arg\min_{\theta\in\Theta}\hat{\boldsymbol{\Omega}}(\theta) \xrightarrow{p} \theta_0$.
   Answer: notes