Econ 501 Final — Fall 2024

Assume i.i.d. data unless otherwise indicated. You are allowed one piece of letter size paper with writing on one side only. Show your work throughout.

1. Provide one pro and one con for each of the Wald test, the score test, and the likelihood ratio test.

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2. Consider the distribution with *distribution* function $F(x) = \{1 - \exp[-x^2/(2\theta_0^2)]\}\mathbb{1}(x \geq 0)$, where $\theta_0 > 0$.

   (a) Derive the formula for the maximum likelihood estimator $\hat{\theta}$ of θ_0.

   (b) Derive the asymptotic distribution of $\hat{\theta}$ as a function of θ_0 only (i.e. without moments of the distribution F). *Hint:* It may be helpful to know that $A_p = \int_0^\infty t^p \exp(-t^2/2)\,\mathrm{d}t$ satisfies $A_0 = \sqrt{\pi/2}$, $A_1 = 1$, and $A_p = (p-1)A_{p-2}$ for $p \geq 2$.

3. Suppose $\bar{\mathbf{x}}$ is the sample mean of an i.i.d. mean zero, (positive) finite variance, sequence. Rank the following objects in terms of their rate of convergence (fastest first): (a) $\bar{\mathbf{x}}$; (b) $\bar{\mathbf{x}}^2$; (c) $\exp(-\bar{\mathbf{x}}) - 1$; (d) $\exp(-\bar{\mathbf{x}}^2) - 1$; (e) $\exp(-1/\bar{\mathbf{x}}^2)$; (f) $\mathbb{1}(|\bar{\mathbf{x}}| \geq \epsilon)$ for fixed $0 < \epsilon < \infty$. Indicate if multiple choices converge equally fast.

4. For each of the six statistics in question 3, determine whether the stated conditions are sufficient to conclude that the statistic, call it $\hat{\theta}$, converges in probability to $\lim_{n\to\infty} \mathbb{E}\hat{\theta}$. Explain briefly.

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5. Suppose that the distribution of $\mathbf{y}_i \mid \mathbf{x}_i = x$ is symmetric around $\theta_0^\intercal x$ and that $\mathbb{E}(\mathbf{x}_i \mathbf{x}_i^\intercal) \geq 0$. Suppose that you were to estimate $\theta_0 \in \Theta = [-100, 100]^d$ using

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \log[1 + (\mathbf{y}_i - \theta^\intercal \mathbf{x}_i)^2].$$

   (a) Establish consistency of $\hat{\theta}$ for $\theta_0 = \arg\min_{\theta \in \Theta} \mathbb{E} \log[1 + (\mathbf{y}_i - \theta^\intercal \mathbf{x}_i)^2]$.

(b) Derive the asymptotic distribution of $\hat{\theta}$ assuming that all elements of θ_0 are less than 100 in absolute value.

6. *(bonus question)* Which country has the highest population per capita?

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6. (a) Suppose that $\{\mathbf{x}_i\}$ is an independent sequence of random variables such that each $\mathbf{x}_i$ has support $[-bi, bi]$ for some $2 < b < \infty$ and i the index of $\mathbf{x}_i$. Suppose further that $\mathbb{E}\mathbf{x}_i = i$ and that $\mathbb{V}\mathbf{x}_i = 2i$. What is the slowest diverging sequence $\{C_n\}$ for which $\sum_{i=1}^{n}(\mathbf{x}_i - i) = O_p(C_n)$?

(b) Using only the information given and without assuming that the $\mathbf{x}_i$'s are independent or identically distributed, show that $\forall i\,:\,\mathbb{E}\mathbf{x}_i = 2$ and $\mathbb{P}(\min_i \mathbf{x}_i \geq 1) = 1$ implies that $2\mathbb{P}(\sum_{i=1}^{100} \log \mathbf{x}_i > 200) \leq 1$.

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