Please write your name here: *This exam has six pages.*

Econ 501 Midterm 1 — October 8

You are allowed one piece of letter size paper with writing on one side only. Show your work throughout.

1. Suppose that $\mathbf{x}, \mathbf{y}$ are random variables with the same distribution and finite variance. Define $\mathbf{a} = \mathbf{x} - \mathbf{y}$ and $\mathbf{b} = \mathbf{x} + \mathbf{y}$. Are $\mathbf{a}, \mathbf{b}$ (a) independent? (b) uncorrelated? (c) mutually conditionally mean independent, i.e. $\mathbb{P}[\mathbb{E}(\mathbf{a} \mid \mathbf{b}) = \mathbb{E}\mathbf{a}] = 1$ and $\mathbb{P}[\mathbb{E}(\mathbf{b} \mid \mathbf{a}) = \mathbb{E}\mathbf{b}] = 1$? Please explain briefly.

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2. Suppose that $\mathbf{x}$ has an $E(1)$ (standard exponential) distribution. Recall that the exponential distribution with parameter λ has density function $f(t) = \lambda \exp(-\lambda t)\mathbb{1}(t > 0)$ and that for a standard exponential, $\mathbb{E}\mathbf{x}^p = p!$ for all integer $p > 0$.

   (a) Determine all cumulants of the standard exponential distribution.

   (b) Determine the distribution function of $\mathbf{z} = \log \mathbf{x}$.

   (c) Now suppose that $\mathbf{y} \mid \mathbf{x} \sim E(1/\mathbf{x})$. Determine $\mathbb{V}[\mathbb{V}(\mathbf{y} \mid \mathbf{x})]$.

3. (a) Consider the sequence of density functions $\{f_n\}$ with $f_n(x) = (n+1)n^{-2}x^{(1/n)-1}(1-x)\mathbb{1}(0 \leq x \leq 1)$. Is $\int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = 1$? Prove or disprove.

(b) Let $g_n(x) = \log(x^n + 1)$. Is $\int_0^1 \lim_{n\to\infty} g_n(x)\,\mathrm{d}x = 0$? Prove or disprove.

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(c) Let μ, ν be measures defined on the Borel Field on $\mathbb{R}$ with for any $a \leq b$, $\mu([a,b]) = b^3 - a^3$ and $\nu([a,b]) = \Phi(b) - \Phi(a) - b\phi(b) + a\phi(a)$ where Φ, ϕ are the standard normal distribution function and density function respectively. Determine the Radon Nikodym derivative of ν with respect to μ.

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4. Let $\Omega = \{0, 1, \ldots, 100\}$. Let $A = \{0, 1, \ldots, 20\}$, $B = \{0, 2, 4, \ldots, 100\}$, $C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$ (the prime numbers in Ω), $D = \{0, 5, \ldots, 100\}$, $E = \{0, 3, 6, \ldots, 99\}$. Suppose that we pick an element of Ω such that each element has equal probability of being selected. Determine $\mathbb{P}[C \cap (D \cup E^c)^c \mid A \cap B]$.

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5. Consider the probability space $(\Omega, \mathcal{B}, \lambda)$ with $\Omega = [0,1]$, $\mathcal{B}$ the Borel field, and λ the Lebesgue measure.

   (a) Consider the function $\mathbf{x} : \Omega \to \mathbb{R}$ with

   $$\mathbf{x}(\omega) = \begin{cases} \omega, & \text{if } \omega \text{ is rational,} \\ 1 - \omega^2, & \text{if } \omega \text{ is irrational .} \end{cases}$$

   Show that $\mathbf{x}$ is a random variable.

   (b) Determine $\mathbb{E}\mathbf{x}$.

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