Econ 501 Midterm 1 — October 14, 2025

You are allowed one piece of letter size paper with writing on one side only. Show your work throughout. The questions are roughly ordered in increasing order of difficulty.

1. Consider the function $f_{\mathbf{xy}}(x, y) = [1 + \alpha(1 - 2x)(1 - 2y)]\mathbb{1}(0 \le x \le 1)\mathbb{1}(0 \le y \le 1)$.

   (a) For which values of α is $f_{\mathbf{xy}}$ a density function?
   Answer: $|\alpha| \le 1$

   (b) For those values of α, determine the conditional distribution function $F_{\mathbf{y}|\mathbf{x}}(y|x)$ for all values of $x, y \in [0, 1]$.
   Answer: $y + \alpha(1 - 2x)y(1 - y)$

2. Consider a random variable $\mathbf{x}$ with density function $f_{\mathbf{x}}(x) = \mathbb{1}(x > 0) \exp[-\pi(\log x)^2]/x$. Determine the density function of $\mathbf{y} = 1/\mathbf{x}$.
   Answer: The same.

3. For each of the following statements, either show that the statement is true or provide a counter example.

   (a) Only the first two cumulants of the normal distribution can be nonzero.
   Answer: True, the cumulant generating function is quadratic.

   (b) For any random variables $\mathbf{x}, \mathbf{y}$ with finite second moments, $\mathbb{P}[\mathbb{V}(\mathbf{y} \mid \mathbf{x}) \le \mathbb{V}\mathbf{y}] = 1$.
   Answer: False, take e.g. $\mathbf{x}$ Bernoulli with $p = 1/2$ and $\mathbf{y}|\mathbf{x} = x \sim \mathcal{N}(0, x + 1)$.

   (c) If $\mathbf{x}$ and $\mathbf{y}$ both have a normal distribution and are uncorrelated then $\mathbf{x}$ and $\mathbf{y}$ must be independent.
   Answer: False, take $f_{\mathbf{xy}}(x, y) = \phi(x)\phi(y)[1 + v(x)v(y)]$ for any odd function v bounded by one.

4. Let $\ulcorner x \urcorner$ denote the smallest integer no less than x and $\llcorner x \lrcorner$ the largest integer no greater than x. Let μ be such that for any $a \le b$,

$$\mu\big([a, b]\big) = \begin{cases} [\exp(1 - \ulcorner a \urcorner) - \exp(-\llcorner b \lrcorner)] \;/\; (e - 1), & \ulcorner a \urcorner \ge 0, \\ 0, & b < 0, \\ [1 - \exp(-\llcorner b \lrcorner)] \;/\; (e - 1), & \ulcorner a \urcorner < 0 \le b. \end{cases}$$

   Let further ψ be such that for any $a \le b$, $\psi\big([a, b]\big) = 2^{\llcorner b+1 \lrcorner} - 2^{\ulcorner a \urcorner}$. Determine the Radon-Nikodym derivative of μ with respect to ψ if both are measures defined on the Borel field on $\mathbb{R}$.

Answer: $(2e)^{-x}$ for $x \geq 0$

5. Let $g(t) = \int_0^\infty \exp(-x^2)\cos(tx)\,dx$. Show that $g'(t) = -\int_0^\infty x\exp(-x^2)\sin(tx)\,dx$. *Hint:* Recall that $g'(t) = \lim_{n\to\infty}\{n[g(t+1/n) - g(t)]\}$.

   Answer: Apply dominated convergence. Note that $h_n(t) = n[g(t+1/n) - g(t)]$ is bounded, uniformly in n.