Econ 501 Midterm 2 — November 19

You are allowed one piece of letter size paper with writing on one side only. Show your work throughout.

1. (a) Suppose that $\{(\mathbf{x}_i, \mathbf{y}_i)\}$ is an i.i.d. sequence of random vectors with $\mu_x = 1,\ 0 < \mu_y < 2$, and $0 < \sigma_x^2, \sigma_y^2 < \infty$. Assuming that $\mathbf{x}_i, \mathbf{y}_i$ are independent, determine the limit distribution of $\sqrt{n}(\bar{\mathbf{x}}^{\bar{\mathbf{y}}} - 1)$. Make sure you get explicit expressions for the mean and variance.

$$\sqrt{n}\begin{pmatrix}\bar{x}-\mu_x\\ \bar{y}-\mu_y\end{pmatrix} \xrightarrow{d} N\left(0, \begin{pmatrix}\sigma_x^2 & 0\\ 0 & \sigma_y^2\end{pmatrix}\right)$$

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(b) Characterize the limit distribution of $n \log^2 \bar{\mathbf{x}} - \sqrt{n}(\bar{\mathbf{y}} - \mu_y)$ under the same conditions.

$$\underline{n(\log \bar x)^2 - \sqrt{n}(\bar y - \mu_y)}$$

$$\left[\sqrt{n}(\log \bar x - \log \mu_x)\right]^2 - \sqrt{n}(\bar y - \mu_y)$$

Delta method: $\sqrt{n}(\log \bar x - \log 1) \xrightarrow{d} N(0, \sigma_x^2)$.

z, z' i.i.d. w/ $N(0,1)$.

$$[\quad]^2 - \sqrt{n}(\quad) \xrightarrow{d} \underline{\sigma_x^2 z^2} \underline{- \sigma_y^2 z'}$$

2. True or false? Explain briefly. No points without a correct explanation.

(a) If $\mathbf{x}$ is a random variable with finite second moments, $\mathbf{y}$ is independent of $\mathbf{x}$ with $\mathbb{P}(\mathbf{y} = 1) = \mathbb{P}(\mathbf{y} = -1) = 1/2$, and $\mathbf{z} = \mathbf{xy}$, then $\mathbf{x}$ and $\mathbf{z}$ are necessarily uncorrelated.

(b) $\mathbf{x}_n = O_p(n^{-1/2}) \Rightarrow \mathbf{x}_n = o_p(1)$.

$x_n = O_p(\frac{1}{\sqrt{n}})$.

$\forall \varepsilon > 0$. $\exists K$. s.t. $Pr(|\sqrt{n} \cdot x_n| > K) < \varepsilon$, $\forall n > N$.

$\Leftrightarrow Pr(|x_n| > \frac{K}{\sqrt{n}}) < \varepsilon$.

$x_n = o_p(1)$. Take any $\delta > 0$. choose N^*. s.t. $\frac{K}{\sqrt{n}} < \delta$. $\forall n > N^*$.

$\forall \varepsilon > 0$. $\delta > 0$, $Pr(|x_n| < \delta) \leq Pr(|x_n| > \frac{K}{\sqrt{n}}) < \varepsilon$, for all $n > N^*$.

3. Suppose that $\mathbf{x} \sim N(1, 1)$ and $\mathbf{y} \sim N(0, 1)$ are joint normal with correlation ρ. Determine $\mathbb{E}(\mathbf{y} \mid |\mathbf{x}| \leq 1)$ as a function of ρ.

$\Rightarrow O_p\left(\frac{1}{\sqrt{n}}\right) = o_p(1)$

4. Suppose $\bar{\mathbf{x}}$ is the sample mean of an i.i.d. mean zero, (positive) finite variance, sequence. Rank the following objects in terms of their rate of convergence (fastest first): (a) $\bar{\mathbf{x}}$; (b) $\bar{\mathbf{x}}^2$; (c) $\exp(-\bar{\mathbf{x}}) - 1$; (d) $\exp(-\bar{\mathbf{x}}^2) - 1$; (e) $\exp(-1/\bar{\mathbf{x}}^2)$; (f) $\mathbb{1}(|\bar{\mathbf{x}}| \geq \epsilon)$ for fixed $0 < \epsilon < \infty$. Indicate if multiple choices converge equally fast.

$\mathbb{1}(|\bar{x}| \geq \varepsilon) = o_p(r_n^{-1})$ for any r_n.

To see this, $\Pr[r_n \mathbb{1}(|\bar{x}_n| \geq \varepsilon) > \delta] < \Pr(\mathbb{1}(|\bar{x}_n| \geq \varepsilon) > \delta) \to 0$

take $r_n = e^{n}$

(a) $\sqrt{n}(\bar{x} - 0) \xrightarrow{d} N(0, \sigma^2)$.

Delta method

(b) $\bar{x}^2 = O_p\left(\frac{1}{\sqrt{n}}\right)$

$\cancel{(2x)}$

$(x^2)' \to 2x$.

$\exp(-\bar{x}) - 1 = O_p\left(\frac{1}{\sqrt{n}}\right)$

$\exp(-\bar{x}^2) - 1 = O_p\left(\frac{1}{\sqrt{n}}\right)$

if $\forall \varepsilon > 0$. $\Pr\left(r_n \exp\left(-\frac{1}{\bar{x}^2}\right) > k\right) < \varepsilon$. for some $k > 0$. for n large.

$= \Pr\left[\exp\left(-\frac{1}{\bar{x}^2}\right) > \frac{k}{r_n}\right)$

$= \Pr\left(-\frac{1}{\bar{x}^2} > \ln k - \ln r_n\right)$

then. $\exp\left(-\frac{1}{\bar{x}^2}\right) = O_p\left(\frac{1}{r_n}\right)$

$= \Pr\left(\bar{x}^2 > \frac{1}{\ln r_n - \ln k}\right)$

$= \Pr\left(n\bar{x}^2 > \frac{n}{\ln r_n - \ln k}\right)$

5. (a) Suppose $\bar{\mathbf{x}}$ is the sample mean of an i.i.d. mean zero sequence for which $\mathbb{E}|\mathbf{x}_i|^p < \infty$ for some $1 \leq p < \infty$. What convergence rate can you obtain for $\bar{\mathbf{x}}$ as a function of p? Express using O_p, o_p notation.

$$\Pr(r_n \bar{x}_n > k) \leq k^{-p} E\left((r_n \bar{x})^p\right)$$

$$= k^{-p} \left(\frac{r_n}{n}\right)^p E(S_n^p)$$

$$\leq \begin{cases} \left(\frac{r_n}{n}\right)^p k^{-p} \cdot n E(|x_i|^p) & \text{for } 1 \leq p \leq 2 \\ n^{p/2} \left(\frac{r_n}{n}\right)^p k^{-p} E(|x_i|^p) & \text{for } p > 2. \end{cases}$$

(M-Z inequality)

Choose k to be very large.

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(b) Suppose $\bar{\mathbf{x}}$ is the sample mean of an i.i.d. mean zero sequence where $\mathbf{x}_i$ has support $[-1, 1]$. Obtain an upper bound for $\mathbb{P}[\exp(n\bar{\mathbf{x}}^2) \geq e]$ which holds for any size sample. Tighter bounds yield more points.

$$\Pr\left(\exp(n\bar{x}^2) \geq e\right)$$

$$\Rightarrow \Pr\left(|\bar{x}| \geq \frac{1}{\sqrt{n}}\right)$$

$$= \Pr\left(|S_n| \geq \sqrt{n}\right)$$

$$\leq 2 \cdot \exp\left(-\frac{n}{2n\sigma^2 + 2/3\sqrt{n}}\right)$$

$$\leq 2 \cdot \exp\left(-\frac{n}{2n + 2/3\sqrt{n}}\right) \quad (\sigma^2 \leq 1. \text{ since } x \in [-1,1]).$$

Hence. $X_n = O_p(1)$.

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