501 Old Mid2 hints

November 11, 2025

1 Problem 1

1.1 a

$$\sqrt{n}\begin{bmatrix}\bar{x}-\mu_x\\ \bar{y}-\mu_y\end{bmatrix} \xrightarrow{d} \mathcal{N}(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma_x^2 & 0\\ 0 & \sigma_y^2\end{bmatrix})$$

$$\sqrt{n}(\bar{x}^{\bar{y}}-1) = \sqrt{n}(\mu_y\mu_x^{\mu_y-1})(\bar{x}-\mu_x)+\sqrt{n}\mu_x^{\mu_y}\ln(\mu_x)(\bar{y}-\mu_y)+o_p(1) = \mathcal{N}(0,\mu_y^2\sigma_x^2)+o_p(1)$$

1.2 b

$$\sqrt{n}(\ln(\bar{x})-\ln(1)) \xrightarrow{d} \mathcal{N}(0,\sigma_x^2)$$
$$n\ln^2(\bar{x})-\sqrt{n}(\bar{y}-\mu_y) = \sigma_x^2 Z^2 - \sigma_y Z'$$

where Z, Z' are independent standard normal.

2 Problem 2

2.1 a

$$E(Z) = 0 \quad E(XZ) = 0$$

by law of iterated expectation by conditioning on y first.

2.2 b

Yes.

3 Problem 3

$E(Y|X=x) = \rho(x-1)$

$$\begin{aligned} E[Y||x| \leq 1] &= E[E(Y|X)||X| \leq 1] \\ &= \rho E(X||X| \leq 1) - \rho = \rho E(Z| -2 \leq Z \leq 0) \\ &= -\frac{\phi(0) - \phi(-2)}{0.5 - \Phi(-2)} \end{aligned}$$

where Z is standard normal

4 Problem 4

$\bar{x} = O_p(1/\sqrt{n}), \bar{x}^2 = O_p(1/n), \exp(-\bar{x}) - 1 = O_p(1/\sqrt{n}), \exp(-\bar{x}^2) - 1 = O_p(1/n)$

$$\exp(-\frac{1}{\bar{x}^2}) = O_p(\frac{1}{r_n}) \text{ for } \lim_{n\to\infty} \frac{r_n}{e^n} = 0, \mathbb{1}(|\bar{x}| \geq \epsilon) = O_P(r_n) \text{ for any } r_n$$

5 Problem 5

5.1 a

Apply Marcinkiewics-Zygmund inequality,

$$\bar{x} = \begin{cases} O_p(n^{\frac{1-p}{p}}) & 1 \leq p < 2 \\ O_P(n^{-\frac{1}{2}}) & p > 2 \end{cases}$$

5.2 b

Apply Bernstein's inequality,

$$\begin{aligned} &Pr(\exp(n\bar{x}^2) \geq e) = Pr(|\bar{x}| \geq \frac{1}{\sqrt{n}}) \\ &= Pr(|S_n| \geq \sqrt{n}) \\ &\leq 2\exp(-\frac{2n}{n\sigma^2 + 2\sqrt{n}/3}) \end{aligned}$$