Midterm Exam, Econ 510 **YOUR NAME:** ____________________

Monday 2/23/26, 10:00-11:30am

Patrik Guggenberger, Department of Economics, Penn State

Answer all four questions for a total of 40 points (each question is worth 10 points). Closed book exam, no calculators or cell phones allowed. You can write on the front and back sides of each page. GOOD LUCK!

1. Consider the model $Y_i = \alpha_0 + \beta_0 X_i + U_i$ for $i = 1, ..., n$. Let Z_i be another random variable (that we will call an *instrumental variable*). Suppose (Y_i, X_i, U_i, Z_i) are i.i.d. random variables with finite means and variances. The instrumental variables estimator of β_0 is defined by

$$\widehat{\beta}_{IV,n} = \left(n^{-1} \sum_{i=1}^{n} (Z_i - \overline{Z}_n) X_i \right)^{-1} n^{-1} \sum_{i=1}^{n} (Z_i - \overline{Z}_n) Y_i,$$

where $\overline{Z}_n$ denotes the sample average of the Z_i. Do not use high level theorems (like Theorem 12.1... from the lecture notes) here. Derive the limiting results from scratch.

   (a) Find the probability limit of $\widehat{\beta}_{IV,n}$ under the weakest additional conditions that you can (and state what these conditions are).

   (b) Under what additional conditions is $\widehat{\beta}_{IV,n}$ consistent for the true value β_0?

   (c) Find the asymptotic distribution of $n^{1/2}(\widehat{\beta}_{IV,n} - \beta_0)$ under the conditions above and the additional conditions that you specify in parts a. and b. and any additional conditions that are needed.

   (d) Provide an estimator for the variance of the limiting distribution in c. Show that the proposed estimator is consistent.

   (e) Write down the Wald statistic to test the null hypothesis $H_0 : \beta_0 = 0$ versus $H_1 : \beta_0 \neq 0$. Describe the intuition underlying the construction of that statistic.

   (f) Explain how you would construct an (asymptotically valid) 95% confidence region for β based on a Wald test. Provide a discussion about the precise sense of asymptotic validity used here.

2. Under Assumptions EE3 and CF-NS (stated below) we derived the limiting distribution of the GMM estimator $\widehat{\theta}_n$ with nonsmooth stochastic criterion function, namely

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \to_d N(0, (\Gamma'\Gamma)^{-1}\Gamma' V_0 \Gamma(\Gamma'\Gamma)^{-1}).$$

   (a) Provide estimators for V_0 and Γ and discuss their consistency under appropriate conditions.

   (b) Suppose $X_n \to_p \mu$ for some constant μ. Show that there exists a sequence of positive constants δ_n such that $\delta_n \to 0$ as $n \to \infty$ and $P(|X_n - \mu| > \delta_n) \to 0$ as $n \to \infty$.

Setup and Assumptions for the GMM case with nonsmooth stochastic criterion function:

Let $Q_n(\theta) = ||\bar{g}_n(\theta)||$, where $\bar{g}_n(\theta) = n^{-1}\sum_{i=1}^n g(W_i, \theta)$, and $g(\theta) = Eg(W_i, \theta)$.

Assumption CF-NS: (i) θ_0 is in the interior of Θ.

(ii) $g(\theta)$ is differentiable at θ_0 with $\Gamma = (\partial/\partial\theta')g(\theta_0)$ of full rank $d \leq k$.

(iii) $g(\theta_0) = 0$.

(iv) $\sqrt{n}\bar{g}_n(\theta_0) \to_d N(0, V_0)$.

(v) For every sequence of positive constants $\{\delta_n\}_{n\geq 1}$ that converges to zero,

$$\sup_{\theta\in\Theta, ||\theta-\theta_0||<\delta_n} \sqrt{n}||\bar{g}_n(\theta) - g(\theta) - \bar{g}_n(\theta_0)|| \to_p 0.$$

Assumption EE3: (i) $Q_n(\widehat{\theta}_n) = \inf_{\theta\in\Theta} Q_n(\theta) + o_p(n^{-1/2})$ and (ii) $\widehat{\theta}_n \to_p \theta_0$.

3. Consider a moment condition model $Eg(W_i, \theta) = 0 \in R^k$ for $\theta = \theta_0 \in R^p$ where $k \geq p$. Assume CF and EE2 (the high level assumptions that imply asymptotic normality of EE) are satisfied for the GMM model, *except that* $\Gamma = E(\partial/\partial\theta)g(W_i, \theta_0) \in R^{k\times p}$ has rank $p-1$ with its p-th column being a zero vector. In particular, assume the GMM estimator $\widehat{\theta}_n = (\widehat{\theta}'_{\cdot n}, \widehat{\theta}_{pn})'$ (for simplicity with weighting matrix $A_n = I_k$) is consistent, where $\widehat{\theta}'_{\cdot n}$ denotes the first $p-1$ and $\widehat{\theta}_{pn}$ the p-th component of $\widehat{\theta}_n$, respectively. We will establish that $\widehat{\theta}_{pn} - \theta_{p0} = O_p(n^{-1/4})$, where $\theta_0 = (\theta'_{\cdot 0}, \theta_{p0})'$. Let $\widehat{g}_n(\theta) = \sum_{i=1}^n g(W_i, \theta)/n$. Define[1]

$$\widehat{g}_{\cdot n}(\theta_\cdot) = \widehat{g}_n(\theta_\cdot, \widehat{\theta}_{pn}) \quad \text{and} \quad \widehat{g}_{pn}(\theta_p) = \widehat{g}_n(\theta_{\cdot 0}, \theta_p).$$

   i) Do a mean value expansion of $\widehat{g}_{\cdot n}(\widehat{\theta}_{\cdot n})$ about $\theta_{\cdot 0}$ and a second order Taylor expansion of $\widehat{g}_{pn}(\widehat{\theta}_{pn})$ about θ_{p0}. Combine the two expansions and, using consistency, establish that

   $$\widehat{\theta}_{\cdot n} - \theta_{\cdot 0} = (\widehat{\Gamma}'_{\cdot n}\widehat{\Gamma}_{\cdot n})^{-1}\widehat{\Gamma}'_{\cdot n}[(\widehat{g}_n(\widehat{\theta}_n) - \widehat{g}_n(\theta_0)) - \frac{1}{2}\widehat{B}_{pn}(\widehat{\theta}_{pn} - \theta_{p0})^2] + o_p(n^{-1/2}),$$

   where $\widehat{\Gamma}_{\cdot n} = (\partial/\partial\theta_\cdot)\widehat{g}_n(\overline{\theta}_{\cdot n}, \widehat{\theta}_{pn})$ for some vector $\overline{\theta}_{\cdot n}$ on the line segment joining $\theta_{\cdot 0}$ and $\widehat{\theta}_{\cdot n}$ and $\widehat{B}_{pn} = (\partial^2/\partial\theta_p^2)\widehat{g}_n(\theta_{\cdot 0}, \overline{\theta}_{pn})$ for some $\overline{\theta}_{pn}$ between θ_{p0} and $\widehat{\theta}_{pn}$.

   ii) Conclude that

   $$\widehat{g}_n(\widehat{\theta}_n) = \widehat{g}_n(\theta_0) + P_{\widehat{\Gamma}_{\cdot n}}(\widehat{g}_n(\widehat{\theta}_n) - \widehat{g}_n(\theta_0)) + \frac{1}{2}M_{\widehat{\Gamma}_{\cdot n}}\widehat{B}_{pn}(\widehat{\theta}_{pn} - \theta_{p0})^2 + o_p(n^{-1/2}),$$

   where $P_A = A(A'A)^{-1}A'$ and $M_A = I_k - P_A$ for any matrix A with k rows.

   iii) Use ii) and $||\widehat{g}_n(\widehat{\theta}_n)||^2 \leq ||\widehat{g}_n(\theta_0)||^2$ to conclude that

   $$n^{1/2}(\widehat{\theta}_{pn} - \theta_{p0})^2 O_p(1) + (\frac{1}{4}B'M_\Gamma B + o_p(1))n(\widehat{\theta}_{pn} - \theta_{p0})^4 = O_p(1),$$

   where M_Γ and B denote the probability limits of $M_{\widehat{\Gamma}_{\cdot n}}$ and $\widehat{B}_{pn}$, respectively.

   iv) Assume that $B'M_\Gamma B$ is not the zero matrix. Conclude from iii) that $\widehat{\theta}_{pn} - \theta_{p0} = O_p(n^{-1/4})$. Show that that implies that $\widehat{\theta}_{\cdot n} - \theta_{\cdot 0} = O_p(n^{-1/2})$.

---

[1] We leave out transpose signs for the arguments of $\widehat{g}_n$; e.g. we write $(\theta_\cdot, \widehat{\theta}_{pn})$ rather than $(\theta'_\cdot, \widehat{\theta}_{pn})'$ to simplify notation.

4. Assume $(X_i, Y_i) \in R^2, i = 1, ..., n$, are i.i.d. with distribution F (that may depend on n). Let $\mu_X = E_F X_i$ and $\mu_Y = E_F Y_i$. We are interested in a CI C_n for the parameter
$$\theta = \max\{\mu_X, \mu_Y\}$$
(that may depend on n) that satisfies
$$AsyCS = \lim \inf_{n\to\infty} \inf_{F\in\mathcal{F}} P_F(\theta \in C_n) \geq 1 - \alpha,$$
i.e. the asymptotic confidence size is nonsmaller than the nominal coverage probability, where $P_F(\cdot)$ denotes probability when the true distribution is F and the set of distributions $\mathcal{F}$ is suitably restricted such that WLLNs and CLTs can be applied. Namely, take
$$\begin{aligned}\mathcal{F} = \mathcal{F}_{\delta,M} = \{F\ &: F \text{ is the distribution of a random vector } (X,Y);\\ &E_F||(X,Y)||^{2+\delta} \leq M,\ \delta \leq \sigma_X^2, \sigma_Y^2 \leq M \text{ and } |\rho| \leq 1-\delta\},\end{aligned}$$
where ρ, σ_X^2, and σ_Y^2 denote the correlation between X and Y and the variances of these variables and $M < \infty$ and $\delta > 0$.

Denote by $\overline{X}_n$ and $\overline{Y}_n$ the sample averages and by $\widehat{\sigma}_{Xn}^2$ and $\widehat{\sigma}_{Yn}^2$ the sample variances of the X_is and Y_is, respectively,
$$\overline{X}_n = n^{-1}\sum_{i=1}^n X_i, \quad \widehat{\sigma}_{Xn}^2 = n^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2...$$
Denote by
$$\begin{aligned}C_{\mu_X n} &= [\overline{X}_n - n^{-1/2}\widehat{\sigma}_{Xn}z_{1-\alpha/2}, \overline{X}_n + n^{-1/2}\widehat{\sigma}_{Xn}z_{1-\alpha/2}] \text{ and}\\ C_{\mu_Y n} &= [\overline{Y}_n - n^{-1/2}\widehat{\sigma}_{Yn}z_{1-\alpha/2}, \overline{Y}_n + n^{-1/2}\widehat{\sigma}_{Yn}z_{1-\alpha/2}]\end{aligned}$$
the standard $1-\alpha$ two-sided CI for μ_X and μ_Y, respectively, based on inverting a t-test.

Consider the CI C_n for θ defined as
$$C_{\mu_X n} \text{ if } \overline{X}_n \geq \overline{Y}_n \text{ and defined as } C_{\mu_Y n} \text{ otherwise.}$$

i) Discuss what happens to the coverage probability as $n \to \infty$ for fixed F.

ii) Next define drifting sequences $\gamma_{nh} = (\gamma_{1n}, \gamma_{2n}, \gamma_{3n})$, where $\gamma_{1n} = \mu_{1n} - \mu_{2n}$, $\gamma_{2n} = (\sigma_{Xn}, \sigma_{Yn}, \rho_n)$, $\gamma_{3n} = F_n$, and
$$n^{1/2}\gamma_{1n} \to h_1, \quad \gamma_{2n} \to h_2 = (h_{21}, h_{22}, h_{23})$$
for some $h_1 \in R \cup \{\pm\infty\}$ and $h_2 \in [\sqrt{\delta}, \sqrt{M}]^2 \times [\delta - 1, 1 - \delta]$. Derive the limiting coverage probability under γ_{nh}.

iii) Using the result in ii) find a simple formula for $AsyCS$.