Advanced Microeconomics Theory

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Chapter 1

Choice Theory

The utility-maximization approach to choice has several features that help explain its long and continuing dominance in economic analysis:

- **Normative usefulness.** It ties individual choices to a welfare criterion that can be compared with the government's, which underwrites both policy analysis and the modern democratic emphasis on respecting individual preferences.
- **Positive predictions.** It yields sharp comparative-statics predictions: how choices respond when prices, incomes, or other parameters change.
- **Wide scope.** The same machinery applies across consumption, labor supply, finance, and beyond.
- **Compactness.** Empirical predictions follow from a sparse model: a description of the chooser's objectives and constraints, nothing more.

1.1 Preference-Based Approach

Rational choice theory starts from the idea that individuals have preferences and choose so as to maximize utility, where the utility is itself a representation of those preferences. Our primary task is to formalize what that statement means and to extract from it precise predictions about the pattern of decision-making we should observe.

1.1.1 Preference Relation

Definition 1.1.1: (Weak) Preference Relation

Let X be a set of possible choices. Consider a *weak preference relation* $\succsim$ over the set X, as a binary relationship:

$$x \succeq y \iff \text{“}x \text{ is at least as good as } y\text{”}$$

Remark.

The weak preference relation $\succeq$ implies the associated strict preference relation $\succ$ and indifference relation $\sim$:

- x is *strictly preferred* to y, or $x \succ y$, if $x \succeq y$ but not $y \succeq x$.
- x is *indifferent* to y, or $x \sim y$, if $x \succeq y$ and $y \succeq x$.

With the weak preference relation in hand, we now impose two assumptions that together formalize what we mean by a *rational* preference.

The first is *completeness*: an agent is never undecided when faced with two choices.

Definition 1.1.2: Completeness

A preference relation $\succeq$ on X is *complete*, if for all $x, y \in X$, either $x \succeq y$, or $y \succeq x$, or both.

The second is *transitivity*: an agent's weak preference cannot cycle, except among choices to which the agent is indifferent.

Definition 1.1.3: Transitivity

A preference relation $\succeq$ on X is *transitive*, if whenever $x \succeq y$ and $y \succeq z$, $x \succeq z$.

From completeness and transitivity, we have the following corollaries:

Corollary 1.1.4

1. While the definition of transitivity involves only 3-choice cycles, it also extends to all n-object cycles, i.e., that it implies that for any n choices $x_1, x_2, \cdots, x_n \in X$ such that $x_1 \succeq x_2$, $x_2 \succeq x_3$, ..., $x_{n-1} \succeq x_n$, we must have $x_1 \succeq x_n$. (Hint: use induction on n.)
2. Transitivity of a weak preference relation $\succeq$ implies transitivity of the associated strict preference relation $\succ$ and the indifference relation $\sim$.

Remark.

1. We take the *weak* preference relation $\succsim$ as primitive (rather than the strict relation $\succ$) because completeness is the natural axiom for $\succsim$: with strict preference, we would need a third case for indifference, which is messier.
2. Transitivity is inconsistent with certain "framing effects" that show up in experimental data — e.g., when the order in which options are presented systematically changes the chosen alternative.

Completeness and transitivity together let us formalize rationality.

Definition 1.1.5: Rationality

A preference relation $\succsim$ on X is *rational* if it is both *complete* and *transitive*.

1.1.2 Choice Rule

Given preferences, the agent's *choice rule* on an "opportunity set" $B \subseteq X$ induced by the preference relation $\succeq$ is

$$C_{\succeq}(B) = \{x \in B \mid x \succeq y,\ \forall y \in B\},$$

i.e., the set of items in B the agent likes at least as much as every other alternative in B — the most-preferred options.

Remark.

1. $C_{\succeq}(B)$ may contain more than one element — ties between equally-preferred alternatives are perfectly admissible.

2. $C_{\succeq}(B)$ can also be *empty*. The simplest sufficient condition for non-emptiness is finiteness of B: if B is finite and non-empty, then $C_{\succeq}(B)$ is non-empty.

   - For an example where $C_{\succsim}(B)$ is empty, take the preference relation $x \succeq y \iff x \geq y$, with $X = [0, +\infty)$ and $B = (0, 1)$. The supremum 1 is not in B, so no element of B is most preferred.
   - For infinite choice sets, we will later add technical assumptions (compactness of the choice set, continuity of the preference relation) that guarantee a choice exists.

Proposition 1.1.6

Suppose $\succeq$ is complete and transitive. Then, for every finite non-empty set B,

$$C_{\succeq}(B) \neq \varnothing.$$

Proof for Proposition.

Proceed by mathematical induction on the number of elements of B.

- $|B| = 1$, say $B = \{x\}$.
  - By completeness, $x \succeq x$, so $x \in C_{\succeq}(B)$. $C_{\succeq}(B) \neq \varnothing, \forall |B| = 1$.
- Fix $n \geq 1$ and suppose that for all sets B with exactly n elements, $C_{\succeq}(B) \neq \varnothing$. Next we move on to examine the case of $|B| = n + 1$.
  - Take any B_n with $|B_n| = n$. By the inductive hypothesis $C_{\succeq}(B_n) \neq \varnothing$, so pick some $x^* \in C_{\succeq}(B_n)$. Let $B = B_n \cup \{x_{n+1}\}$.
  - By completeness, we have only two (not mutually-exclusive) possibilities:
    * If $x^* \succeq x_{n+1}$, then by definition $x^* \in C_{\succeq}(B)$, so $C_{\succeq}(B) \neq \varnothing$.

  - ∗ If $x_{n+1} \succeq x^*$. Since $x^* \in C_{\succeq}(B_n)$, by definition, $x^* \succeq y, \forall y \in B_n$. By transitivity, this implies $x_{n+1} \succeq y, \forall y \in B$. Therefore, $x_{n+1} \in C_{\succeq}(B)$, so $C_{\succeq}(B) \neq \varnothing$.

- Hence, for every set B with exact $n+1$ elements, $C_{\succeq}(B) \neq \varnothing$. By the principle of mathematical induction, it follows that for every finite set B that is non-empty, $C_{\succeq}(B) \neq \varnothing$.

1.2 Choice-Based Approach

Much empirical work runs in the opposite direction. Rather than starting from preferences and predicting choices, it observes choices and tries to *rationalize* them: to determine whether the observed choices are compatible with preference maximization and, if so, what they imply about the underlying preferences. Under this approach, the choice rule is the primitive object of the theory, and preferences (if they exist) are derived from it.

Definition 1.2.1: Choice Rule

Let $\mathcal{B}$ be the set of all nonempty subsets of X ($\mathcal{B} = 2^X \backslash \varnothing = \{B \neq \varnothing : B \subset X\}$). A *choice rule* is a function $C : \mathcal{B} \to \mathcal{B}$ with the property that for all $B \in \mathcal{B}$, $C(B) \subseteq B$.

Remark.

- $\mathcal{B}$ is the set of all nonempty **subsets** of X, which means all possible set of available choice(s) the agent is facing. And the choice rule C is a mapping from $\mathcal{B}$ to $\mathcal{B}$, which means that the agent is choosing from a set of available choice(s) to pick his most preferred choice(s), also a subset of X.

- Here we assume that we can see the agent choose from *all* possible subsets of X, and that the agent reports *all* of his optimal choices from a given opportunity set.

The bare definition imposes no structure on either the choice rule or any preference relation behind it. Two questions arise:

- If the rule does come from maximizing some underlying preferences, what can we infer about those preferences from the rule alone?

- Is the rule consistent with the maximization of *some* complete and transitive preference relation — i.e., is it *rationalizable*?

Consider the first question. Suppose the choice rule C is consistent with maximization of some preference relation $\succsim$, i.e., $C(\cdot) = C_{\succsim}(\cdot)$. Then, observing for some $A \subseteq X$ that $y \in A$ and $x \in C(A)$ — x is chosen when y is available — lets us infer $x \succsim y$. This in turn implies that for any $B \subseteq X$ with $x \in B$ and $y \in C(B)$, we must also have $x \in C(B)$: we know $x \succsim y$ and $y \succsim z$ for all $z \in B$, so by transitivity $x \succsim z$ for all $z \in B$. By a symmetric argument we also get $y \in C(A)$. So any rationalizable choice rule must satisfy the following property as a *necessary* condition:

Definition 1.2.2: HARP

A choice function $C : \mathcal{B} \to \mathcal{B}$ satisfies *Houthaker's Axiom of Revealed Preference (HARP)* if, whenever $x, y \in A \cap B$, and $x \in C(A)$ and $y \in C(B)$, we have $x \in C(B)$ and $y \in C(A)$.

In words: if x and y are both available in two different choice problems, and x is chosen from one while y is chosen from the other, then x and y must *both* be chosen in *both* problems. HARP rules out the obvious form of inconsistency in observed choices — picking x over y here and y over x there.

We have just shown HARP is necessary for rationalizability. The striking result is that it is also *sufficient*:

Proposition 1.2.3

Suppose $C : \mathcal{B} \to \mathcal{B}$ is nonempty-valued. Then there exists a rational (complete and transitive) preference relation $\succsim$ on X such that $C(\cdot) = C_{\succsim}(\cdot)$ if and only if C satisfies HARP.

Proof for Proposition.

- "Only if" part: shown above.
- "If" part. Suppose C satisfies HARP. We construct a rational preference relation $\succsim_C$ that rationalizes C.
  - *Define the revealed preference relation.* Set $x \succsim_C y$ iff there exists some $A \subseteq X$ with $y \in A$ and $x \in C(A)$. That is, x is revealed preferred to y whenever the agent picks x from some menu that also contained y.
  - *$\succsim_C$ is complete.* For any $x, y \in X$, the set $C(\{x, y\})$ is non-empty, so it contains x, y, or both. In each case the corresponding ranking $x \succsim_C y$ or $y \succsim_C x$ follows by construction.
  - *$\succsim_C$ is transitive.* Suppose $x \succsim_C y$ and $y \succsim_C z$. The set $C(\{x, y, z\})$ is non-empty, so it contains at least one of x, y, z (the cases are not mutually exclusive, but we just need one):
    * $x \in C(\{x, y, z\})$. Since $z \in \{x, y, z\}$, the definition of $\succsim_C$ gives $x \succsim_C z$ directly.
    * $y \in C(\{x, y, z\})$. From $x \succsim_C y$ there exists A_0 with $y \in A_0$ and $x \in C(A_0)$. Apply HARP with this A_0 and $B = \{x, y, z\}$ (both contain x and y) to get $x \in C(\{x, y, z\})$. This reduces to case 1, so $x \succsim_C z$.
    * $z \in C(\{x, y, z\})$. From $y \succsim_C z$ there exists A_0 with $z \in A_0$ and $y \in C(A_0)$. HARP gives $y \in C(\{x, y, z\})$, reducing to case 2, and hence $x \succsim_C z$.
  - $C(\cdot) = C_{\succsim_C}(\cdot)$. It suffices to show that for all $A \in \mathcal{B}$ and $x \in A$, $x \in C(A) \iff x \succsim_C y$ for all $y \in A$.
    * "$\Rightarrow$" holds directly by the definition of $\succsim_C$.
    * "$\Leftarrow$" Take any $x \in C_{\succsim_C}(A)$, so $x \succsim_C y$ for all $y \in A$. Since $C(A)$ is non-empty, there is some $y_0 \in C(A)$. The definition of $\succsim_C$ gives an A_0 with $x, y_0 \in A_0$ and

$x \in C(A_0)$. Then $x, y_0 \in A_0 \cap A$, with $x \in C(A_0)$ and $y_0 \in C(A)$, so HARP yields $x \in C(A)$.

Remark.

1. HARP does the entire work in the "if" direction: it is precisely the axiom that endows the revealed preference relation $\succsim_C$ with the transitivity needed to call it rational.

2. The rationalizability result requires observing the *whole* choice function C — that is, (i) for any given choice set, all of the agent's optimal choices are observed, not just some of them, and (ii) the agent's optimal choices are observed across *every* choice set. Real data is rarely so generous:

   - In consumer-choice problems, the menus A we see are typically budget sets indexed by prices and income — a strict subcollection of $\mathcal{B}$, not all of $\mathcal{B}$.
   - The first form of incompleteness (only seeing some of the optimal choices from a given menu) is essentially intractable. For the second (only seeing some menus), other revealed-preference axioms have been developed. The *Weak Axiom of Revealed Preference (WARP)* is the natural counterpart of HARP for choice rules restricted to budget sets and required to be single-valued (so the HARP conclusion sharpens to $x = y$). WARP is necessary for rationalizability but not sufficient on budget data alone; the *Generalized Axiom of Revealed Preference (GARP)* closes the gap and is both necessary and sufficient.

Chapter 2

Consumer Theory

Choice theory says that a rational decision-maker picks the most preferred option from her choice set. Consumer theory specializes this framework to the most extensively studied application — a consumer choosing bundles of goods subject to a budget constraint — and uses the *preference-based approach* throughout. The agenda has three parts:

1. **Formalize the choice set and preferences.**
   - Set of alternatives: the *consumption set*.
   - Choice set (within the consumption set): the *budget set*.
   - Preferences: a *utility representation* of the preference relation.
2. **Derive optimal choices** from the budget set and the preference relation.
3. **Analyze the properties** of those optimal choices — i.e., of demand.

2.1 Setups

We start with four assumptions that we will maintain throughout consumer theory:

Assumption 2.1.1

1. *Perfect* information.
2. Consumers are *price takers*.
3. Prices are *linear*.
4. Goods are *divisible*.

Remark.

- *Price-taking* means the consumer treats prices $\boldsymbol{p}$ as known, fixed, and exogenous — no searching for deals, no bargaining for discounts.
- *Linear prices* means the total cost of a bundle is just $\boldsymbol{p} \cdot \boldsymbol{x}$; no quantity discounts or two-part tariffs.

- *Divisibility* is captured by working in $\mathbb{R}^n_+$. This is mostly a mathematical convenience: a discrete-good problem can still be analyzed by restricting attention to the integer points in $\mathbb{R}^n_+$, so the divisibility assumption does not rule out applications to indivisible goods.

For simplicity, we take the consumption set to be the entire non-negative orthant.

Definition 2.1.2: Consumption Set

With n goods, the *consumption set* is given by:

$$X = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \cdots, n\}.$$

The budget set is a subset of the consumption set; it further restricts the bundles the consumer can actually afford given income and prices.

Definition 2.1.3: Budget Set

With n goods and income of m, given a price vector $\boldsymbol{p} = (p_1, \cdots, p_n)' \geq 0, \boldsymbol{p} \neq \boldsymbol{0}$, the *budget set* is given by:

$$B(\boldsymbol{p}, m) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{x} \leq m, \boldsymbol{x} \geq \boldsymbol{0}\}.$$

By reinterpreting the consumption goods and the budget, the derivation of the budget set can be extended to encompass other economic problems. For example, consumption-leisure choice, and inter-temporal choice.

2.2 Utility Representation

Preference relations are intuitive but awkward to work with directly. A utility representation lets us recast preference maximization as ordinary optimization over a real-valued function — a much more tractable object.

Definition 2.2.1: Utility Representation

A preference relation $\succsim$ on X is represented by a *utility function* $u : X \to \mathbb{R}$ if

$$x \succsim y \iff u(x) \geq u(y).$$

This definition makes utility an *ordinal* object: the actual numerical values carry no economic content. Only the implied ranking matters — u encodes the consumer's relative preferences over bundles, nothing more.

If u represents $\succsim$, then the choice rule defined upon budget set is:

$$C(B(\boldsymbol{p}, m); \succsim) = \left\{\boldsymbol{x} \mid \boldsymbol{x} \text{ solves } \max_{\boldsymbol{x} \in B(\boldsymbol{p}, m)} u(\boldsymbol{x})\right\}.$$

An equivalent notation of such choice rule is $C_{\succsim}(B)$.

The natural question is whether *every* rational preference relation admits a utility representation. If the consumption set X is *finite*, the answer is yes — and the construction is direct.

Proposition 2.2.2

If X is finite, then any rational (complete and transitive) preference relation $\succsim$ on X can be represented by a utility function $u : X \to \{1, \cdots, n\}$, where $n = |X|$.

Proof for Proposition.

Proceed by induction on $|X|$.

- *Base case:* $|X| = 1$, say $X = \{x\}$. Set $u(x) = 1$. The condition holds vacuously.
- *Inductive step:* suppose any rational preference on a set of size n admits the desired representation. Take any X with $|X| = n + 1$.
  - Since $C_{\succsim}(X)$ is non-empty (Ch. 1 Proposition on finite choice sets), the complement $Y = X \setminus C_{\succsim}(X)$ has at most n elements. By the inductive hypothesis, the restriction of $\succsim$ to Y admits a utility representation $u : Y \to \{1, 2, \ldots, n\}$.
  - We extend the domain of u to X by setting $u(x) = n + 1$ for each $x \in C_{\succsim}(X)$. By construction, we have $u(x) \in \{1, \cdots, n, n + 1\}$ for all $x \in X$.
  - Now we show that the constructed u represents $\succsim$, i.e., for any $x, y \in X$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Suppose $x \succsim y$. There are three possibilities:
    * $x \in C_{\succsim}(X), y \in Y \iff u(x) = n + 1 \geq u(y)$.
    * $x, y \in C_{\succsim}(X) \iff u(x) = u(y) = n + 1$, also, $u(x) \geq u(y)$.
    * $x, y \in Y$. Then, since by construction u represents $\succsim$ on Y, $x \succsim y$ if and only if $u(x) \geq u(y)$.

■

When X is infinite the situation is more subtle. In general, a rational preference relation on $\mathbb{R}^n_+$ need not admit a utility representation $u : \mathbb{R}^n_+ \to \mathbb{R}$. The standard counter-example is lexicographic preferences.

Example.

Consider the lexicographic preferences on the square $X = \mathbb{R}^2$, where $(x_1, x_2) \succ (y_1, y_2)$ if either (1) $x_1 > y_1$ or (2) $x_1 = y_1$ and $x_2 > y_2$. These preferences cannot be represented by a utility function, and this is also an example for which "indifference curves" do not exist, because the agent is never indifferent between any two choices.

To see this, suppose by contradiction there exists a utility representation $u(x, y)$. For any $x \in \mathbb{R}_+$, consider the interval $I(x) = \big(\inf_y u(x, y), \sup_y u(x, y)\big)$. $I(x)$ is not degenerate, and $I(x_1), I(x_2)$ do not overlap for $x_1 \neq x_2$. Since rational numbers are dense, we construct a function r to pick a rational number inside $I(x)$, i.e., $r(x) \in I(x)$. Since $I(x_1), I(x_2)$ do not overlap, $r(x_1) \neq r(x_2)$ for $x_1 \neq x_2$. Thus, $r : \mathbb{R}_+ \to \mathbb{Q}_+$ is an injective function. Since $\mathbb{R}_+$ is uncountable and $\mathbb{Q}_+$ is countable, this mapping is impossible.

However, note that if we replace $X = \mathbb{R}^2_+$ with $X' = \mathbb{Q}^2_+$, the lexicographic preference relation would admit a utility representation. More generally, this idea is formalized into the proposition in countable set case.

For *countable* X (possibly infinite), every rational preference relation admits a utility representation — and the construction below gives one explicitly.

Proposition 2.2.3

If $X \neq \varnothing$ is countable, then any rational (complete and transitive) preference relation $\succsim$ on X can be represented by a utility function $u : X \to (0, 1)$.

Proof for Proposition.

- First, construct a mapping of utility function $u : X \to (0, 1)$.
  - Let $(x_n)_{n=1}^{\infty}$ be a countable enumeration of X.
  - Let $u(x_1) = \frac{1}{2}$. Consider x_{n+1}, $n \geq 1$.
    - If $x_{n+1} \sim x_i$ for some $1 \leq i \leq n$, then define $u(x_{n+1}) = u(x_i)$.
    - Otherwise, define

$$M_n = \max\{\max\{u(x_i) : x_{n+1} \succ x_i, 1 \leq i \leq n\}, 0\}$$
$$m_n = \min\{\min\{u(x_i) : x_i \succ x_{n+1}, 1 \leq i \leq n\}, 1\}$$

      By construction $M_n > m_n$. Define $u(x_{n+1}) = \frac{M_n + m_n}{2}$.

- Second, prove that $u(\cdot)$ is a utility representation of preference relation $\succsim$.
  - Take any $y, z \in X$. Since X is countable, $\exists i, j$ such that $y = x_i$ and $z = x_j$. Without loss of generality, suppose $i \leq j$.
  - By construction, $u(x_i) = u(x_j)$ if and only if $x_i \sim x_j$. For $x_j \nsim x_i$, $u(x_j) > u(x_i)$ if and only if $x_j \succ x_i$.

■

The pathology of the lexicographic preference relation is a sudden preference reversal: $(3, 3) \succ (3, 2)$, yet $(3, 2) \succ (x, 3)$ for any x strictly less than 3 — the ranking jumps discontinuously as x crosses 3. To rule this out we impose a *continuity* restriction on preferences. Continuity is also desirable from a revealed-preference standpoint: any finite set of observed choices that is consistent with HARP can be rationalized by a continuous preference relation, so continuity costs us nothing empirically.

Definition 2.2.4: Continuity

A preference relation $\succsim$ on $X \subseteq \mathbb{R}^n$ is *continuous* if for any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \to x, y^n \to y$, and $x^n \succsim y^n$ for all n, we have $x \succsim y$.

Continuity gives us more than mere existence of a utility representation — it gives us a *continuous* one.

Proposition 2.2.5

Any complete, transitive and continuous preference relation $\succsim$ on X on $X \subseteq \mathbb{R}^n$ can be represented by a continuous utility function $u : X \to \mathbb{R}$.

Proof for Proposition.

In order to have a simple, constructive proof, we prove the proposition only for the case of a monotone preference relation $\succsim$ on $X = \mathbb{R}^n_+$.

Let $e = (1, \cdots, 1)$ and consider bundles of the form $\alpha e = (\alpha, \cdots, \alpha)$ where $\alpha \geq 0$. For each $x \in \mathbb{R}^n_+$, we construct a utility number as follows: $u(x) = \max A(x)$, where $A(x) = \{\alpha \in R_+ : \alpha e \preceq x\}$. To see that the set $A(x)$ has a maximal point, note that the set is

- Nonempty, since $0 \in A(x)$ by monotonicity of $\succsim$;
- Closed, by the continuity of $\succsim$;
- Bounded, since by monotonicity of $\succsim$, $\alpha \leq \max\{x_1, \cdots, x_n\}$ for each $\alpha \in A(x)$.

Now we show that we must have $u(x)e \sim x$.

1. $u(x)e \preceq x$. This is satisfied by construction of $u(x) \in A(x)$.
2. $u(x)e \succsim x$. For each $n \geq 1$, we have $u(x) + \frac{1}{n} \notin A(x)$, hence $(u(x) + \frac{1}{n})e \not\preceq x$, therefore by completeness of $\succsim$ we have $(u(x) + \frac{1}{n})e \succsim x$, which by continuity of $\succsim$ implies
$$\lim_{n\to\infty} (u(x) + \frac{1}{n})e = u(x)e \succsim x.$$

Now it remains to show that the constructed utility function $u(\cdot)$ has

1. Ability to represent the preference relation $\succsim$, and
2. Continuity.

For representation part, note that by transitivity $x \succsim y$ if and only if $u(x)e \succsim u(y)e$ (since $u(x)e \sim x \succsim y \sim u(y)e$), and by monotonicity of $\succsim$, this holds if and only if $u(x) \geq u(y)$. For continuity part, this is more subtle and not covered here.

Remark.

1. The construction in the proof specifies the utility of any bundle x by finding the point on the 45° line on the indifference curve passing through x.
   - This specification is, of course, completely arbitrary; just for mathematical convenience.
   - To reflect this arbitrariness, utility representation of preferences is *ordinal*, i.e., only the induced preference ordering of choices is meaningful, instead of the exact utility numbers assigned to them.
   - In fact, if u represents $\succsim$, then $U(\cdot) = v(u(\cdot))$ also represents $\succsim$ so long as $v : \mathbb{R} \to \mathbb{R}$ is an increasing function.

2. The thought process and the introduction of the additional assumption are the most important for this part.

   - Recall that our motivation is to come up with a tractable framework to analyze the consumer's problem. One possibility is to attach a utility level to each consumption bundle (ordinal utility representation).
   - We ask whether any preference relation can be represented by a utility function. The answer is yes, if the consumption set X is finite (or countable), but no if $X = \mathbb{R}^n_+$. We notice that the problem with the counter-example (e.g., the lexicographic preference relation) is that there are sudden preference reversals.
   - We then impose the restriction of "continuity" of preferences to rule out those unfavorable scenarios and show that any rational and continuous preference relation on $X = \mathbb{R}^n_+$ can be represented by a continuous utility function.

3. The continuity of preference relation and continuity of its utility representation is not interdependent. If a preference relation can be represented by a continuous utility function, then such preference relation must be continuous. On the other hand, a continuous preference relation can be represented by a discontinuous utility function, if you like. (That is not convenient for mathematical issues though.)

2.3 Utility Maximizing Problem

What distinguishes consumer theory from the general choice-theoretic framework is the specific structure of the consumer's choice set — it is determined by prices and wealth. This structure is what lets us derive economically meaningful comparative statics. With a utility representation $u(\cdot)$ of $\succsim$ in hand, the consumer's optimization is the *Utility Maximization Problem* (UMP), also called the *Consumer Problem* (CP).

Definition 2.3.1: Utility Maximization Problem

$$\begin{aligned} &\max_{\boldsymbol{x}\in\mathbb{R}^n_+} u(\boldsymbol{x}) \\ \text{s.t. } &\boldsymbol{p}\cdot\boldsymbol{x} \le m \\ &x_i \ge 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

That is: given prices $\boldsymbol{p} = (p_1, \ldots, p_n)' \ge \mathbf{0}$, $\boldsymbol{p} \ne \mathbf{0}$ and wealth m, the consumer picks a non-negative bundle $\boldsymbol{x} = (x_1, \ldots, x_n)'$ to maximize utility subject to spending no more than m. Equivalently, the CP can be written as:

$$C(B(\boldsymbol{p}, m); \succsim) = \left\{ \boldsymbol{x} \;\middle|\; \boldsymbol{x} \text{ solves } \max_{\boldsymbol{y}\in B(\boldsymbol{p},m)} u(\boldsymbol{y}) \right\}.$$

2.3.1 Existence of Optimal Choice(s)

The first question is whether the UMP has any solution. Under mild conditions, it does.

Proposition 2.3.2

Suppose that the consumer has a rational (complete and transitive) and continuous preference relation and makes rational decisions, then for $(\boldsymbol{p}, m) \gg \mathbf{0}$, they have (at least) one optimal choice.

Proof for Proposition.

- Since the consumer has a complete, transitive and continuous preference relation, then her preferences can be represented by a continuous utility function $u(\cdot)$.
- When $(\boldsymbol{p}, m) \gg \mathbf{0}$, the budget set $B(\boldsymbol{p}, m)$ is closed and bounded and hence compact.
- A continuous function on a compact set has at least one maximizer.

2.3.2 Further Assumptions

Existence is guaranteed under rationality, continuity, and $(\boldsymbol{p}, m) \gg \mathbf{0}$. The remaining questions are practical:

- How do we actually *find* the optimal choice(s)?
- Which additional restrictions on $\succsim$ simplify the analysis?

Locally Non-Satiation One useful simplification is to replace the budget inequality with the equality $\boldsymbol{p} \cdot \boldsymbol{x} = m$. Intuitively, if the consumer always thinks "more is better," she should never leave money on the table — at any optimum, she exhausts her budget.

Definition 2.3.3: Monotonicity

A preference relation $\succsim$ on $X = \mathbb{R}^n_+$ is *monotone* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$, we have:

- $\boldsymbol{x} \geq \boldsymbol{y} \Longrightarrow \boldsymbol{x} \succsim \boldsymbol{y}$;
- $\boldsymbol{x} \gg \boldsymbol{y} \Longrightarrow \boldsymbol{x} \succ \boldsymbol{y}$.

Monotonicity is more than we need; the weaker condition of *local non-satiation* already does the job.

Definition 2.3.4: Locally Non-Satiation

A preference relation $\succsim$ on $X = \mathbb{R}^n_+$ is *locally non-satiated* if for any $\boldsymbol{x} \in X$ and $\varepsilon > 0$, there exists $\boldsymbol{y} \in Ball(\boldsymbol{x}, \varepsilon) \cap X$ such that $\boldsymbol{y} \succ \boldsymbol{x}$.

Intuitively, local non-satiation rules out any *bliss point*: from any bundle, the consumer can do strictly better with an arbitrarily small change. Note that monotonicity implies local non-satiation (but not conversely).

Proposition 2.3.5

Suppose that the consumer has a complete, transitive, continuous and locally non-satiated preference relation and makes rational decisions, then for $(\boldsymbol{p}, m) \gg \mathbf{0}$, the budget constraint must hold with equality at any optimal choice $\boldsymbol{x}^*$, i.e., $\boldsymbol{p} \cdot \boldsymbol{x}^* = m$.

Proof for Proposition.

Intuitively, if the consumer does not exhaust her budget, then there must be a nearby affordable bundle which is strictly more preferred, by locally non-satiation.

- Consider any bundle $\boldsymbol{x} \in X = \mathbb{R}^n_+$ where $\boldsymbol{p} \cdot \boldsymbol{x} < m$. Since the preference relation $\succsim$ is locally non-satiated, for any $\varepsilon > 0$, there exists $\boldsymbol{y} \in Ball(\boldsymbol{x}, \varepsilon) \cap X$ such that $\boldsymbol{y} \succ \boldsymbol{x}$.

- We claim that $\boldsymbol{y} \in B(\boldsymbol{p}, m)$, i.e., the alternative bundle falls within the consumer's budget set for $\varepsilon > 0$ small enough, so that $\boldsymbol{x}$ cannot be an optimal choice.

  Let $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z}$. By construction, $||\boldsymbol{z}|| < \varepsilon$. In particular, $|z_i| < \varepsilon$, for any $i = 1, 2, \cdots, n$. It follows that $\boldsymbol{p} \cdot \boldsymbol{y} = \boldsymbol{p} \cdot \boldsymbol{x} + \boldsymbol{p} \cdot \boldsymbol{z} \leq \boldsymbol{p} \cdot \boldsymbol{x} + n \cdot \varepsilon \cdot \max_i p_i$. As ε take appropriately small values, we finish our proof.

■

Convexity Another possible simplification is the case of *unique* optimal choice. Notice that the consumer's budget set $B(\boldsymbol{p}, m)$ is convex. Based on this, when the utility function $u(\cdot)$ is *strictly quasi-concave*, there is a *unique* global maximizer. This translates into the following condition of strictly convex preference relations.

Definition 2.3.6: Convexity

A preference relation $\succsim$ on $X = \mathbb{R}^n_+$ is *convex* if for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ such that $\boldsymbol{x} \succsim \boldsymbol{z}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$, we have $t\boldsymbol{x} + (1-t)\boldsymbol{y} \succsim \boldsymbol{z}$, for any $t \in [0, 1]$. If $t\boldsymbol{x} + (1-t)\boldsymbol{y} \succ \boldsymbol{z}$ for any $\boldsymbol{x} \neq \boldsymbol{y}$ and $t \in (0, 1)$, then the preference relation is *strictly convex.*

Remark.

1. Convexity can be equivalently defined as: For any $\boldsymbol{x}, \boldsymbol{y} \in X$ such that $\boldsymbol{y} \succsim \boldsymbol{x}$, we have $t\boldsymbol{y} + (1-t)\boldsymbol{x} \succsim \boldsymbol{x}$ for any $t \in [0, 1]$.

2. An equivalent way to describe convexity involves indifference curve and *upper contour set* of choice bundles, Upper Contour Set of $y = \{x \in X : x \succsim y\}$, graphically the area sitting upper-right above the indifference curve (included). Convexity of preferences amounts to the assumption that the upper contour set of any $y \in X$, is a *convex* set.

3. Convexity is fundamental in the standard model of competitive economics. When consumer preferences are convex, market clearing prices exist; otherwise this may not exist. Convexity is also needed to be able to recover consumer preferences from choices from various budget sets. Convexity is often described as capturing the idea that the agent like diversity. However, whether convexity makes sense often depends on the interpretation of the goods space, in particular on the level of aggregation (e.g., over time or categories).

Proposition 2.3.7

Suppose that the consumer has a complete, transitive, continuous and strictly convex preference relation and makes rational decisions, then for $(\boldsymbol{p}, m) \gg \mathbf{0}$, there is exactly one optimal choice.

Proof for Proposition.

- The rational and continuous preference relation has guaranteed the existence of at least one optimal choice.
- Suppose to the contrary that the consumer has at least two optimal choices $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, and $\boldsymbol{x}^* \neq \boldsymbol{y}^*$, then by optimality $\boldsymbol{x}^* \sim \boldsymbol{y}^*$. Construct a bundle $\mathbf{w} = \frac{1}{2}\boldsymbol{x}^* + \frac{1}{2}\boldsymbol{y}^*$. Since the consumer's preference relation is strictly convex, $\boldsymbol{w} \succ \boldsymbol{x}^* \sim \boldsymbol{y}^*$. Moreover, since the budget set $B(\boldsymbol{p}, m)$ is convex when $(\boldsymbol{p}, m) \gg \mathbf{0}$, so $\mathbf{w} \in B(\boldsymbol{p}, m)$. So neither $\boldsymbol{x}^*$ nor $\boldsymbol{y}^*$ can be an optimal choice, which is a contradiction.

These assumptions on $\succsim$ each have a direct counterpart on any representing utility function u. The translation makes it easier to work in the utility space and import the right structural property without re-deriving it.

Proposition 2.3.8

Suppose the preference relation $\succsim$ on X can be represented by $u : X \to \mathbb{R}$. Then,

1. $\succsim$ is monotone if and only if u is non-decreasing.
2. $\succsim$ is locally non-satiated if and only if u has no local maxima in X.
3. $\succsim$ is (strictly) convex if and only if u is (strictly) quasi-concave.

2.3.3 Marshallian Demand & Indirect Utility Function

Definition 2.3.9: Indirect Utility Function

Given $(\boldsymbol{p}, m) \gg \mathbf{0}$, the *indirect utility function* is defined to give the optimal utility level:

$$v(\boldsymbol{p}, m) = \sup_{\boldsymbol{x} \in B(\boldsymbol{p}, m)} u(\boldsymbol{x}).$$

Here we rigorously use "sup" to define $v(\boldsymbol{p}, m)$ instead of "max", because sometimes $u(\boldsymbol{x})$ does not behave well to have a maximum.

Definition 2.3.10: Marshallian Demand Correspondence

The *Marshallian demand correspondence* is defined as the consumer's optimal choice(s):

$$\boldsymbol{x}^M(\boldsymbol{p}, m) = \{\boldsymbol{x} \in B(\boldsymbol{p}, m) : u(\boldsymbol{x}) = v(\boldsymbol{p}, m)\}.$$

In general $\boldsymbol{x}^M(\boldsymbol{p}, m)$ is a *set* of utility-maximizing bundles — it collapses to a single point only under strict convexity (see below).

Throughout this section assume the consumer has a rational preference relation $\succsim$ and faces $(\boldsymbol{p}, m) \gg \mathbf{0}$. The Marshallian demand and indirect utility function inherit the following properties from $\succsim$:

Proposition 2.3.11: Properties of Marshallian Demand Correspondence and Indirect Utility Function

- **Existence of optimal choice(s)**: If $\succsim$ is continuous, then the Marshallian demand $\boldsymbol{x}^M(\boldsymbol{p}, m) \neq \varnothing$.
- **Structure of Marshallian demand**: If $\succsim$ is convex, then $\boldsymbol{x}^M(\boldsymbol{p}, m)$ is a convex set. If $\succsim$ is strictly convex, then $\boldsymbol{x}^M(\boldsymbol{p}, m)$ Is a singleton.
- **Homogeneity**: Both $v(\boldsymbol{p}, m)$ and $\boldsymbol{x}^M(\boldsymbol{p}, m)$ are homogeneous of degree 0 in $(\boldsymbol{p}, m)$, that is, for any $t > 0$, $v(t\boldsymbol{p}, tm) = v(\boldsymbol{p}, m)$ and $\boldsymbol{x}^M(t\boldsymbol{p}, tm) = \boldsymbol{x}^M(\boldsymbol{p}, m)$.
- **Monotonicity of $v(\boldsymbol{p}, m)$**: $v(\boldsymbol{p}, m)$ is non-increasing in $\boldsymbol{p}$ and non-decreasing in m. If $\succsim$ is locally non-satiated, then $v(\boldsymbol{p}, m)$ is strictly increasing in m.
- **Walras' Law**: If $\succsim$ is locally non-satiated, then $\boldsymbol{p} \cdot \boldsymbol{x} = m$, for any $\boldsymbol{x} \in \boldsymbol{x}^M(\boldsymbol{p}, m)$.

2.3.4 Derivation of Utility Maximization Problem

The consumer's utility maximization problem is

$$\begin{aligned} &\max_{\boldsymbol{x} \in \mathbb{R}^n_+} u(\boldsymbol{x}) \\ \text{s.t. } &\boldsymbol{p} \cdot \boldsymbol{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

For $(\boldsymbol{p}, m) \gg \mathbf{0}$, the constraint qualification is always satisfied, and we can apply the necessary KKT conditions when $u(\cdot)$ is continuously differentiable.
The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu) = u(\boldsymbol{x}) + \lambda \left(m - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \mu_i x_i,$$

where λ is the Lagrange multiplier on the budget constraint and, for each i, μ_i is the multiplier on the constraint that $x_i \geq 0$. The UMP is then transformed into:

$$\max_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \lambda, \mu) = \max_{\boldsymbol{x}} u(\boldsymbol{x}) + \lambda \left(m - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \mu_i x_i.$$

The first-order conditions are given by:

- w.r.t. x_i : $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i + \mu_i = 0$.
- Inequality constraints: $m - \sum_{i=1}^{n} p_i x_i \geq 0, x_i \geq 0, \lambda \geq 0, \mu_i \geq 0$.
- Complementary slackness: $\lambda \left(m - \sum_{i=1}^{n} p_i x_i \right) = 0, \mu_i x_i = 0$.

Remark.

- In contrast to equality constraints, the direction of each inequality constraint determines the way in which we set up the Lagrangian.

  Intuitively, we make sure each multiplier is non-negative and penalize constraint violations. For instance, when the budget constraint is violated, that is, $m - \sum_{i=1}^{n} p_i x_i < 0$, the value of the Lagrangian strictly decreases.

- Despite little economic meaning, λ represents the *shadow price* of wealth, that is, marginal utility of an additional unit of income.

  However, note that utility only has ordinal meanings. Nothing in the consumer theory developed so far suggests any basis for using the shadow price of wealth to guide redistribution policies.

- If the preference relation is well-behaved (i.e., locally non-satiated and strictly convex) and the non-negativity constraints are not binding, then $\frac{\partial u}{\partial x_i} = \lambda p_i$, and we are back to the familiar "tangency conditions", that is, for all i, j:

$$MRS_{ij} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} = \frac{p_i}{p_j}.$$

  The "tangency conditions" say that at the consumer's maximum, relative marginal utility of any two choices equals their relative price, that is, all possible inner "gains from trade" has been realized (thus no more inner "gain from trade").

To facilitate the derivation of the consumer's problem, we should **first check whether the preference relation is locally non-satiated and strictly convex**.

- Case 1: If *locally non-satiated* and *strictly convex*, then we proceed in two steps
  1. Directly apply the "tangency conditions": $\frac{MU_i}{p_i} = \frac{MU_j}{p_j} = \lambda$, for any i, j.
  2. Check the *non-negativity constraints* and apply the complementary slackness condition(s) if necessary.

     Note that in the previous step we first assume that the non-negativity holds and that we have interior solution.

- Case 2: If neither locally non-satiated nor strictly convex, then use logic or economic intuition to tackle the problem.

Cobb-Douglas Utility Example.

Suppose that the consumer's preference relation can be represented by the following utility function:

$$u(x_1, x_2, x_3) = (x_1 + a)(x_2 + b)(x_3 + c)$$

where $a, b, c \geq 0$ are non-negative constants. Moreover, the consumer faces constant prices $p_1, p_2, p_3\mathbf{0}$ and has income $m \geq 0$.

1. First suppose that $a = b = c = 0$. Solve for the consumer's Marshallian demand correspondence $\boldsymbol{x}^M(x_1, x_2, x_3)$ and indirect utility function $v(p_1, p_2, p_3, m)$.
2. Next suppose that $a, b, c > 0$. Solve for the consumer's Marshallian demand correspondence $\boldsymbol{x}^M(p_1, p_2, p_3, m)$ and indirect utility function $v(p_1, p_2, p_3, m)$.

Claim: Cobb-Douglas Utility

Cobb-Douglas utility representation for a preference relation $\succsim$ can be written as

$$u(x_1, x_2, \cdots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Then the consumer would spend her income on each good according to its share,

$$x_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} \cdot \frac{m}{p_i} \iff p_i x_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} \cdot m$$

Solution.

1. To solve the first question, we need to rigorously follow the steps:
   - Step 1:Apply the positive monotonic transformation for computation-convenience: $v = \ln u$.
   - Step 2: Check the locally non-satiation and convexity of preference relation for simplification of optimal solution.
     - Check that the preference relation is monotonic and hence locally non-satiated.
     - Check that the preference relation is strictly convex, that is, the utility function is strictly quasi-concave.
   - Step 3: Now that both non-satiation and convexity are satisfied, apply the "tangency conditions":
     $$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \frac{MU_3}{p_3}$$
   - Step 4: Together with the budget constraint, so we have the Marshallian demand correspondence $\boldsymbol{x}^M(\boldsymbol{p}, m)$.

$$\boldsymbol{x}^M(p_1, p_2, p_3, m) = \left(\frac{m}{3p_1}, \frac{m}{3p_2}, \frac{m}{3p_3}\right) \geq \mathbf{0}.$$

2. The first three steps are quite similar to the previous part and thus omitted. In step 4, this time we have

$$\boldsymbol{x}_1^M(p_1,p_2,p_3,m) = \frac{m+p_1a+p_2b+p_3c}{3p_1} - a$$
$$\boldsymbol{x}_2^M(p_1,p_2,p_3,m) = \frac{m+p_1a+p_2b+p_3c}{3p_2} - b$$
$$\boldsymbol{x}_3^M(p_1,p_2,p_3,m) = \frac{m+p_1a+p_2b+p_3c}{3p_3} - c$$

Note that depending on the parameter values, we may or may not have $\boldsymbol{x}^M(\boldsymbol{p},m) \geq \mathbf{0}$. In other words, the non-negativity constraints may be binding and are for us to check. For simplicity, suppose that

$$2p_1a - p_2b - p_3c \geq 2p_2b - p_1a - p_3c \geq 2p_3c - p_1a - p_2b$$

The other five symmetric cases are similar. In our assumption, $p_1a \geq p_2b \geq p_3c$.

(a) $m \geq 2p_1a - p_2b - p_3c$, then the non-negativity constraints are not binding, and we have the interior solution described above.

(b) $2p_1a - p_2b - p_3c \geq m$, then at optimum, $x_1^* = 0$. The constraint $x_1^M \geq 0$ is binding, so the consumer only purchases goods 2 and 3, and we have

$$\boldsymbol{x}^M(p_1,p_2,p_3,m) = \left(0, \frac{m+p_2b+p_3c}{2p_2} - b, \frac{m+p_2b+p_3c}{2p_3} - c\right)$$

i. $m \geq p_2b - p_3c$, then $\boldsymbol{x}^M$ is as above.

ii. $p_2b - p_3c > m$, then the two constraints $x_1^M \geq 0$ and $x_2^M \geq 0$ are both binding, so the consumer only purchases good 3, and

$$\boldsymbol{x}^M(p_1,p_2,p_3,m) = \left(0, 0, \frac{m}{p_3} - c\right)$$

Inter-Temporal Choice Example.

You have a saving $s > 0$ to spend for this year and next year. Since you are now in graduate school, you will not earn any additional income over the two years. Suppose your utility is time-separable and is given by $v(c_1,c_2) = u(c_1) + \beta u(c_2)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable function with $u'(c) > 0$ and $u''(c) < 0$ for any $c \in \mathbb{R}_+$ and $0 < \beta < 1$ is the discount factor. Further suppose that the going interest rate is $0 < r < 1$, which will remain constant. Let c_1^* and c_2^* be your optimal consumption choices for this year and next year.

1. Give a (necessary and sufficient) condition for $(c_1^*, c_2^*) \gg \mathbf{0}$.

2. Suppose that $(c_1^*, c_2^*) \gg \mathbf{0}$, compare c_1^* with c_2^*.

Solution.

1. The utility maximization problem is given by:

$$\begin{aligned} &\max_{c_1,c_2} v(c_1,c_2) = u(c_1) + \beta u(c_2) \\ \text{s.t.} \ \ & c_1 + \frac{c_2}{1+r} \leq s \\ & c_1, c_2 \geq 0 \end{aligned}$$

   Since $u(\cdot)$ is strictly increasing and strictly concave, $v(\cdot)$ is also strictly increasing and strictly concave (in particular, strictly quasi-concave), and we can directly apply the "tangency condition":

$$\frac{u'(c_1)}{1} = \frac{\beta u'(c_2)}{\frac{1}{1+r}}$$

   By strictly monotonicity, $c_1 + \frac{c_2}{1+r} = s$. For $(c_1^*, c_2^*) \gg \mathbf{0}$, we need:

$$\frac{u'(s)}{u'(0)} < \beta(1+r) < \frac{u'(0)}{u'\left((1+r)s\right)}$$

2. At the optimum,

$$\frac{u'(c_1)}{u'(c_2)} = \beta(1+r)$$

   - If $\beta(1+r) > 1$, then $\frac{u'(c_1)}{u'(c_2)} > 1$ and $c_1^* < c_2^*$.
   - If $0 < \beta(1+r) < 1$, then $\frac{u'(c_1)}{u'(c_2)} < 1$ and $c_1^* > c_2^*$.
   - If $\beta(1+r) = 1$, then $\frac{u'(c_1)}{u'(c_2)} = 1$ and $c_1^* = c_2^*$.

   Intuitively, $\beta(1+r)$ represents how your tradeoff between next year and this year compares with that of the market when $c_1 = c_2$.

Utility Representation Example.

Let $X = \mathbb{R}_+ \times \mathbb{N}$, where (x,t) is interpreted as receiving x yuan at time t. Consider the following six properties of preference relations on X:

- Rationality (completeness and transitivity).
- Continuity.
- There is indifference between receiving 0 yuan at time 0 and receiving 0 yuan at any other time.
- It is (strictly) better to receive any positive amount of money as soon as possible.
- Money is always desirable.
- The preference between (x,t) and $(y,t+1)$ is independent of t.

Consider the following questions:

1. Use precise mathematical language to formally define the six properties.

2. Suppose a preference relation on X can be represented by the utility function $v(x,t) = u(x)\beta^t$, where $0 < \beta < 1$ and $u(\cdot)$ is continuous, strictly increasing and $u(0) = 0$. Check whether this preference relation satisfies each of the six properties.

3. Suppose a preference relation on X satisfies all of the six properties. Show that this preference relation must admit a utility representation.

4. Use precise mathematical language to formalize the idea that "one preference is more patient than another".

5. Based on your definition in part (4), prove or disprove the following statement: A preference relation represented by $v_1(x,t) = u_1(x)\beta_1^t$ is more patient than another preference relation represented by $v_2(x,t) = u_2(x)\beta_2^t$ if $0 < \beta_2 < \beta_1$ (where $u_1(\cdot)$ and $u_2(\cdot)$ are both continuous, strictly increasing and $u_1(0) = u_2(0) = 0$).

Solution.

1. Mind yourself that the time t is not continuous here, so take caution when you try to define a "limit" with regard to t.

   - Continuity: For any $t, t' \in \mathbb{N}$, and any pair of sequences $\{x(n)\}_{n=1}^{\infty}$ and $\{y(n)\}_{n=1}^{\infty}$ from $\mathbb{R}^+$ with $x(n) \to x^*$, $y(n) \to y^*$. If $(x(n), t) \succsim (y(n), t')$ for all n, we have $(x^*, t) \succsim (y^*, t')$.
   - The preference between (x,t) and $(y, t+1)$ is independent of t: For any $x, y \in \mathbb{R}^+$, and $t, t' \in \mathbb{N}$, we have $(x,t) \succsim (y, t+1) \iff (x, t') \succsim (y, t'+1)$.

2. A continuous preference relation can be represented by a discontinuous function. However, if a preference relation can be represented by a continuous function, then the preference relation must be continuous.

3. Recall the proof in our lecture of existence of utility representation for any rational and continuous preference relation.

   - Claim 1: For any pair (x,t), there is a unique number $u(x,t) \in \mathbb{R}^+$ such that $(x,t) \sim (u(x,t), 0)$.

     Proof: First, by indifference when receiving nothing, we have $(0,t) \sim (0,0)$. For any pair (x,t), we have $(x,t) \succsim (0,0)$ and $(x+1,t) \succsim (x,t)$. Then by continuity, there is y for which $(x,t) \sim (y,0)$, and we then define $u(x,t) := y$.
   - Claim 2: The preference relation is represented by $u(x,t)$.

     By claim 1 and property that money is more desirable:

$$u(x,t) \geq u(y,t') \iff (u(x,t),0) \succsim (u(y,t'),0) \iff (x,t) \succsim (y,t')$$

4. The definition should be clearly based on preference relations and try to be somewhat math-irrelevant.

   $\succsim_1$ is more patient than $\succsim_2$ if for any (x,t) and any (y,t') with $t' > t$, $y > x$:

$$(y,t') \succsim_2 (x,t) \Longrightarrow (y,t') \succsim_1 (x,t)$$

The definition means that if I prefer to wait from t to t' under $\succsim_1$, then I will also prefer to wait from t to t' when I'm more patient (say under $\succsim_2$).

5. Intuitively, if the two preference relations value money differently at the baseline level (i.e., simply in terms of money), they would generate different preference over combinations of money and receiving time.

Example.

Let $\succsim$ be a rational (complete and transitive) preference relation on $X = \mathbb{R}^2_+$. Consider the following three properties:

- Additivity: If $(x_1, x_2) \succsim (y_1, y_2)$, then for any t, s such that $(x_1+t, x_2+s), (y_1+t, y_2+s) \in \mathbb{R}^2_+$, $(x_1+t, x_2+s) \succsim (y_1+t, y_2+s)$.
- Strong monotonicity: If $x_1 \geq y_1$ and $x_2 \geq y_2$, then $(x_1, x_2) \succsim (y_1, y_2)$. If in addition, $x_1 > y_1$ or $x_2 > y_2$, then $(x_1, x_2) \succ (y_1, y_2)$.
- Standard continuity: For any two sequences $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ and $\{\boldsymbol{y}_n\}_{n=1}^{\infty}$, if $\boldsymbol{x}_n \succsim \boldsymbol{y}_n$ for any n, and $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^*$ and $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}^*$, then $\boldsymbol{x}^* \succsim \boldsymbol{y}^*$.

Consider the following questions:

1. Show that if $\succsim$ has a linear utility representation, i.e., $u(x_1, x_2) = ax_1 + bx_2$, for some $a, b > 0$, then this preference relation satisfies the above three properties.
2. Show that these three properties are necessary for the preference relation $\succsim$ to have a linear utility representation, i.e., show that for any pair of the three properties, there is a preference relation that does not satisfy the third property.
3. Show that if $\succsim$ satisfies the three properties, then this preference relation admits a linear utility representation, i.e., there exists $a, b > 0$ such that $u(x_1, x_2) = ax_1 + bx_2$, for any $(x_1, x_2) \in \mathbb{R}^2_+$. (Hint: Think about the indifference curves/sets of this preference relation.)

Solution.

1. Easy to verify. Notice that if a preference relation can be represented by a continuous function, then the preference relation must be continuous.
2. This question means the three properties are "parallel" from a preference relation to have a linear utility representation.
   - (i)(ii)$\overset{\times}{\Longrightarrow}$(iii): The lexicographic preference: $(x_1, x_2) \succsim (y_1, y_2)$ if either $x_1 > x_2$ or $x_1 = x_2$ and $y_1 > y_2$.
   - (i)(iIi)$\overset{\times}{\Longrightarrow}$(ii): $u(x_1, x_2) = x_1 - x_2$; $u(x_1, x_2) = x_1 - \frac{1}{x_2}$.
   - (iI)(iIi)$\overset{\times}{\Longrightarrow}$(i): $u(x_1, x_2) = x_1^2 + x_2^2$; $u(x_1, x_2) = x_1$.
3. Starting from possible intuitions from linear utility representation, we need to establish the following two properties of the indifference curve:

- Property 1: The indifference curves are linear.
- Property 2: The indifference curves are parallel, downward sloping and not thick.

In order to establish the two properties, we first prove the following two lemmas:

- Lemma 1: For $\boldsymbol{x} \neq \boldsymbol{y}$, if $\boldsymbol{x} \sim \boldsymbol{y}$, then for $\boldsymbol{z} = \frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}$ and $\boldsymbol{z}' = 2\boldsymbol{y} - \boldsymbol{x}$, we have $\boldsymbol{x} \sim \boldsymbol{y} \sim \boldsymbol{z} \sim \boldsymbol{z}'$.
- Lemma 2: For $\boldsymbol{x} \neq \boldsymbol{y}$, if $\boldsymbol{x} \sim \boldsymbol{y}$, then for any $\boldsymbol{w} = t\boldsymbol{x} + (1-t)\boldsymbol{y}, 0 \leq t \leq 1$, we have $\boldsymbol{x} \sim \boldsymbol{y} \sim \boldsymbol{w}$.

Here we omit the technical proofs and move on with establishment of property 1. Pick any point on the horizontal axis $\boldsymbol{x} = (x, 0), x > 0$. By the proof of the utility representation in lecture and strong monotonicity, $\exists 0 < w < x$ such that $\boldsymbol{x} \sim \boldsymbol{w} = w\boldsymbol{e} = (w, w)$. Connect $\boldsymbol{x}$ and $\boldsymbol{w}$ and extend it to the vertical axis. Denote the intersection of the ray $\boldsymbol{xw}$ with the vertical axis as $\boldsymbol{y} = (0, y)$. Jointly from Lemma 1 and 2 we can say the points on the line $\boldsymbol{xy}$ are indifferent to each other. Finally, by strong monotonicity, the indifference curves must be downward sloping and for any $x \neq x'$, we cannot have $(x, 0) \sim (x', 0)$. Moreover, if $(x, 0) \sim (w, w)$, then for any $t \geq -w$, we have $(x + t, 0) \sim (w + t, w)$, so the indifference curves are parallel.

2.4 Expenditure Minimization Problem

To separate the substitution effect from the income effect of a price change, we introduce a dual problem: minimize expenditure subject to achieving a fixed utility level. This is the *Expenditure Minimization Problem* (EMP), and it will prove indispensable for the Slutsky decomposition later in this chapter.

Definition 2.4.1: Expenditure Minimization Problem

$$\begin{aligned} &\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \\ \text{s.t. } &u(\boldsymbol{x}) \geq u \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

Definition 2.4.2: Expenditure Function; Hicksian Demand Correspondence

Let $F(\boldsymbol{p}, u) = \{\boldsymbol{x} \in \mathbb{R}^n_+ : u(\boldsymbol{x}) \geq u\}$ be the feasible set. Define the optimal (minimal) value function as the *expenditure function*:

$$e(\boldsymbol{p}, u) = \inf_{\boldsymbol{x} \in F(\boldsymbol{p}, u)} \boldsymbol{p} \cdot \boldsymbol{x},$$

and the consumer's optimal choice(s) as the *Hicksian demand correspondence*:

$$\boldsymbol{x}^H(\boldsymbol{p}, u) = \{\boldsymbol{x} \in F(\boldsymbol{p}, u) : \boldsymbol{p} \cdot \boldsymbol{x} = e(\boldsymbol{p}, u)\}.$$

Because the utility level is held fixed, a price change leaves the consumer's welfare unchanged by construction. The resulting movement in Hicksian demand is therefore a pure ***substitution effect*** *— no income effect contaminates it.*

As we did for the UMP, we approach the EMP from two angles:

- When can we simplify the problem (e.g., existence of solution(s), binding utility level and uniqueness of solution)?
- Properties of expenditure function and Hicksian demand correspondence.

2.4.1 Existence of Solution

Proposition 2.4.3

Suppose $u(\cdot)$ represents a continuous preference relation and that $\boldsymbol{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$, then the expenditure minimization problem has at least one minimizer, i.e.,

$$\boldsymbol{x}^H(\boldsymbol{p}, u) \neq \varnothing.$$

Proof for Proposition.

- Pick any $\boldsymbol{x}_0 \in F(u)$ and consider the alternative feasible set:

  $$\tilde{F} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : u(\boldsymbol{x}) \geq u \text{ and } \boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{x}_0\}.$$

  It is easy to see that the expenditure minimization problem with the feasible set $\tilde{F}$ has the same solution as the original problem. Moreover, the key motivation for this construction is that $\tilde{F}$ is closed and bounded, and hence compact.

  - Boundedness is crafted by picking $\boldsymbol{x}_0$ purposefully but without loss of generality.
  - Closedness is guaranteed by continuity of $u(\cdot)$.
    - Note that $u(\cdot)$ need not be continuous, because we can always find an alternative continuous function $v(\cdot)$ that represents the same preference relation regardless of the continuity of $u(\cdot)$. Thus, $\tilde{F}$ is closed.
    - Note that the continuity of preference relation has no direct relation to the continuity of its utility representation.

- The objective function $\boldsymbol{p} \cdot \boldsymbol{x}$ is continuous. We know that a continuous function on a compact set has at least one minimizer, which ends our proof.

■

2.4.2 Binding Utility Level

Proposition 2.4.4

Suppose $u(\cdot)$ represents a continuous preference relation and that $\boldsymbol{p} \gg \mathbf{0}$, $u \geq u(\mathbf{0})$ and $F(u) \neq \varnothing$, then at any minimizer $\boldsymbol{x}^*$,

$$u(\boldsymbol{x}^*) = u.$$

Proof for Proposition.

- By the argument in the previous proposition, we can assume without loss that $u(\cdot)$ is continuous. Suppose to the contrary that a minimizer $\boldsymbol{x}^*$, we have $u(\boldsymbol{x}^*) > u$. Since $u \geq u(\mathbf{0})$, $\boldsymbol{x}^* \neq \mathbf{0}$.

- By the continuity of $u(\cdot)$, we know for some $\varepsilon > 0$. $u\left((1-\varepsilon)\boldsymbol{x}^*\right) > u$. It follows that $(1-\varepsilon)\boldsymbol{x}^* \in F(u)$ and that $\boldsymbol{p} \cdot (1-\varepsilon)\boldsymbol{x}^* < \boldsymbol{p} \cdot \boldsymbol{x}^*$, which is a contradiction to the optimality of $\boldsymbol{x}^*$.

■

Remark.

The binding condition here in EMP is much weaker than that in UMP, where we do not put "any" additional condition on preference relation. One can understand this as the objective function in EMP is itself locally non-satiated.

2.4.3 Unique Minimizer

Proposition 2.4.5

Suppose $u(\cdot)$ represents a continuous and strictly convex preference relation and that $\boldsymbol{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$. Then the expenditure minimization problem has exactly one minimizer, i.e., $\boldsymbol{x}^H(\boldsymbol{p}, u)$ is a singleton.

Proof for Proposition.

- By the preceding proposition, *at least one minimizer exists.*

- Suppose to the contrary that there are two minimizers $\boldsymbol{x}^* \neq \boldsymbol{y}^*$. Then by feasibility, $u(\boldsymbol{x}^*) \geq u$ and $u(\boldsymbol{y}^*) \geq u$. Since the preference relation is strictly convex, $u(\cdot)$ is strictly quasi-concave, so for the bundle $\mathbf{w} = \frac{1}{2}\boldsymbol{x}^* + \frac{1}{2}\boldsymbol{y}^* \neq \mathbf{0}$, we have $u(\mathbf{w}) > \min\{u(\boldsymbol{x}^*), u(\boldsymbol{y}^*)\} \geq u$. By continuity, for some small $\varepsilon > 0$, $u((1-\varepsilon)\mathbf{w}) \geq u$. Moreover, $\boldsymbol{p} \cdot (1-\varepsilon)\mathbf{w} = \frac{1}{2}(1-\varepsilon)\boldsymbol{p} \cdot \boldsymbol{x}^* + \frac{1}{2}(1-\varepsilon)\boldsymbol{p} \cdot \boldsymbol{y}^* < \boldsymbol{p} \cdot \boldsymbol{x}^* = \boldsymbol{p} \cdot \boldsymbol{y}^*$, which is a contradiction.

■

2.4.4 Summary of Properties

Proposition 2.4.6: Properties of Expenditure Function and Hicksian Demand Correspondence

Suppose $\boldsymbol{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$.

- **Existence of minimizer**: If $u(\cdot)$ represents a continuous preference relation and that, then the Hicksian demand $\boldsymbol{x}^H(\boldsymbol{p}, u) \neq \varnothing$.
- **Structure of Hicksian demand**: If $u(\cdot)$ represents a convex preference relation, then $\boldsymbol{x}^H(\boldsymbol{p}, u)$ is a convex set. If $u(\cdot)$ represents a continuous and strictly convex preference relation, then $\boldsymbol{x}^H(\boldsymbol{p}, u)$ is a singleton.
- **Homogeneity**: $e(\boldsymbol{p}, u)$ is homogeneous of degree 1 in $\boldsymbol{p}$, that is, for any $t > 0$, $e(t\boldsymbol{p}, u) = te(\boldsymbol{p}, u)$.
- **Monotonicity of $e(\boldsymbol{p}, u)$**: $e(\boldsymbol{p}, u)$ is non-decreasing in $\boldsymbol{p}$ and u. If $u(\cdot)$ represents a continuous preference relation, then $e(\boldsymbol{p}, u)$ is strictly increasing in u when $u \geq u(\mathbf{0})$.
- **Binding utility level**: Suppose $u(\cdot)$ represents a continuous preference relation and $u \geq u(\mathbf{0})$. Then at any minimizer $\boldsymbol{x}^*$, $u(\boldsymbol{x}^*) = u$.

Since $\min \boldsymbol{p} \cdot \boldsymbol{x}$ is equivalent to $\max -\boldsymbol{p} \cdot \boldsymbol{x}$, EMP can be solved in an analogous manner to UMP. EMP and UMP share the same "tangency condition" for interior solutions:

$$\frac{MU_i}{MU_j} = \frac{p_i}{p_j}$$

Example.

Suppose a consumer's preference relation can be represented by the following utility function:

$$u(x_1, x_2) = \ln x_1 + x_2.$$

Moreover, the consumer faces constant prices $(p_1, p_2) \gg \mathbf{0}$ and has income $m > 0$.

1. Solve the consumer's utility maximization problem to derive the Marshallian demand $\boldsymbol{x}^M(p_1, p_2, m)$ and indirect utility function $v(p_1, p_2, m)$.
2. Solve the consumer's expenditure minimization problem to derive the Hicksian demand $\boldsymbol{x}^H(p_1, p_2, u)$ and expenditure function $e(p_1, p_2, u)$.

Solution.

1. It is easy to check that the preference relation is monotonic and strictly convex. The utility miximization can then be simplified as:

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) = \ln x_1 + x_2 \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m.$$

The "tangency condition" is: $|MRS| = \frac{MU_1}{MU_2} = \frac{1}{x_1} = \frac{p_1}{p_2}$.

Together with the binding budget constraint, we have $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = \frac{m}{p_2} - 1$.

Note that the constraint $x_2 \geq 0$ may be binding. Marshallian demand is given by

$$\boldsymbol{x}^M(p_1, p_2, m) = \begin{cases} \left(\frac{p_2}{p_1}, \frac{m}{p_2} - 1\right), & \text{if } m \geq p_2 \\ \left(\frac{m}{p_1}, 0\right), & \text{if } 0 < m < p_2 \end{cases}$$

2. Similar to the previous part, the expenditure minimization problem can be simplified as:

$$\min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2 \quad \text{s.t. } u(x_1, x_2) = \ln x_1 + x_2 = u.$$

The "tangency condition" is $|MRS| = \frac{MU_1}{MU_2} = \frac{1}{x_1} = \frac{p_1}{p_2}$.

Together with binding utility level, we have $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = u - \ln \frac{p_2}{p_1}$.

Notice the constraint $x_2 \geq 0$ may be binding, so the Hicksian demand is given by:

$$\boldsymbol{x}^H(p_1, p_2, m) = \begin{cases} \left(\frac{p_2}{p_1}, u - \ln \frac{p_2}{p_1}\right), & \text{if } u \geq \ln \frac{p_2}{p_1} \\ (e^u, 0), & \text{if } u < \ln \frac{p_2}{p_1} \end{cases}$$

Remember to check if the utility function is monotone and quasi-concave beforehand!

2.5 Duality and Comparative Statics

2.5.1 Duality between UMP and EMP

Recall consumer's utility maximization problem (UMP):

$$\begin{aligned} &\max_{\boldsymbol{x}} u(\boldsymbol{x}) \\ \text{s.t. } &\boldsymbol{p} \cdot \boldsymbol{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

Consumer's expenditure minimization problem (EMP):

$$\begin{aligned} &\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \\ \text{s.t. } &u(\boldsymbol{x}) \geq u \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

The two problems swap the roles of objective and constraint: the UMP's objective $u(\boldsymbol{x})$ becomes the EMP's constraint, and the UMP's constraint $\boldsymbol{p} \cdot \boldsymbol{x}$ becomes the EMP's objective. In optimization language they are *dual problems*, and the duality forces a tight relationship between Marshallian and Hicksian demand.

Proposition 2.5.1

Suppose $u(\cdot)$ is a utility function that represents a continuous and locally non-satiated preference relation on $X = \mathbb{R}^n_+$, then for any $\boldsymbol{p} \gg \mathbf{0}$, we have:

1. For any $m \geq 0$, $\boldsymbol{x}^M(\boldsymbol{p}, m) = \boldsymbol{x}^H(\boldsymbol{p}, v(\boldsymbol{p}, m))$ and $e(\boldsymbol{p}, v(\boldsymbol{p}, m)) = m$.
2. For any $u \geq u(\mathbf{0})$, $\boldsymbol{x}^H(\boldsymbol{p}, u) = \boldsymbol{x}^M(\boldsymbol{p}, e(\boldsymbol{p}, u))$ and $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$.

Proof for Proposition.

- Fix $m > 0$ and any $\boldsymbol{x}_0^M \in \boldsymbol{x}^M(\boldsymbol{p}, m)$, we have:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, m)) \leq \boldsymbol{p} \cdot \boldsymbol{x}_0^M = m.$$

  Fix any $u \geq u(\mathbf{0})$ and any $\boldsymbol{x}_0^H \in \boldsymbol{x}^H(\boldsymbol{p}, u)$, we have

$$v(\boldsymbol{p}, e(\boldsymbol{p}, u)) \geq u(\boldsymbol{x}_0^H) = u.$$

- Applying the first inequality to the wealth level $m = e(\boldsymbol{p}, u)$, we have:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, e(\boldsymbol{p}, u))) \leq e(\boldsymbol{p}, u).$$

  On the other hand, since the preference relation is continuous, $e(\boldsymbol{p}, u)$ is strictly increasing in u, so from the second inequality, we have

$$e(\boldsymbol{p}, v(\boldsymbol{p}, e(\boldsymbol{p}, u))) \geq e(\boldsymbol{p}, u).$$

- Again by strict monotonicity of $e(\boldsymbol{p}, u)$ in u, we have $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$. Similarly, $e(\boldsymbol{p}, v(\boldsymbol{p}, m)) = m$.
- Finally, since the preference relation is continuous and locally non-satiated, the budget constraint must bind at the UMP and the utility level must bind at the EMP. Correspondingly,

$$\begin{aligned} \boldsymbol{x}^M(\boldsymbol{p}, m) &= \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{p} \cdot \boldsymbol{x} = m, u(\boldsymbol{x}) = v(\boldsymbol{p}, m)\} = \boldsymbol{x}^H(\boldsymbol{p}, v(\boldsymbol{p}, m)) \\ \boldsymbol{x}^H(\boldsymbol{p}, u) &= \{\boldsymbol{x} \in \mathbb{R}^n_+ : u(\boldsymbol{x}) = u, \boldsymbol{p} \cdot \boldsymbol{x} = e(\boldsymbol{p}, u)\} = \boldsymbol{x}^M(\boldsymbol{p}, e(\boldsymbol{p}, u)) \end{aligned}$$

Intuitively: pick a wealth level m and let $u = v(\boldsymbol{p}, m)$ be the utility the consumer actually attains; then the Marshallian demand at $(\boldsymbol{p}, m)$ and the Hicksian demand at $(\boldsymbol{p}, u)$ pick out the same bundles. The dual problems describe the same optimum from two angles.

2.5.2 Envelope Theorem

Theorem 2.5.2: Envelope Theorem for Unconstrained Optimization

Let $f : X \times [\underline{\theta}, \bar{\theta}] \to \mathbb{R}$ and $V(\theta) = \sup_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \theta)$. Suppose $f(\boldsymbol{x}, \cdot)$ is differentiable in θ for all $\boldsymbol{x} \in X$. Moreover, there exists an integrable function $b : [\underline{\theta}, \bar{\theta}] \to \mathbb{R}_+$ such that $|f_\theta(\boldsymbol{x}, \theta)| \leq b(\theta)$ for all $\boldsymbol{x} \in X$ and almost all $\theta \in [\underline{\theta}, \bar{\theta}]$. Then $V(\cdot)$ is absolutely continuous and hence differentiable almost everywhere. In addition, for any $\boldsymbol{x}^* \in \arg\max_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \theta)$,

$$V'(\theta) = f_2(\boldsymbol{x}^*(\theta), \theta).$$

Remark.

- The intuition is that, only the direct effect matters; the indirect effect of $\boldsymbol{x}^*(\theta)$ on $V(\theta)$ can be ignored, since $f(\cdot)$ does not change with $\boldsymbol{x}$ at the optimum.
- May get some inspiration from the simplest version.
  - Suppose x is one-dimensional and that the optimizer is unique and differentiable in θ. Combined with F.O.C., we would have:

$$V'(\theta) = f_1(x^*(\theta), \theta) \cdot (x^*)'(\theta) + f_2(x^*(\theta), \theta) = f_2(x^*(\theta), \theta).$$

  - For the envelope theorem to hold, $\boldsymbol{x}$ need not be one-dimensional, $f(\cdot, \theta)$ need not be differentiable in $\boldsymbol{x}$, and the optimal $\boldsymbol{x}^*(\theta)$ need not be unique or differentiable in θ.

Theorem 2.5.3: Envelope Theorem for Constrained Optimization

Suppose X is compact and convex. Let $f, g : X \times \left[\underline{\theta}, \bar{\theta}\right] \to \mathbb{R}$ and $V(\theta) = \sup_{\boldsymbol{x} \in X : g(\boldsymbol{x}, \theta) \geq 0} f(\boldsymbol{x}, \theta)$. The Lagrangian is given by $\mathcal{L}(\boldsymbol{x}, \theta; \lambda) = f(\boldsymbol{x}, \theta) + \lambda g(\boldsymbol{x}, \theta)$. Suppose f and g are continuous and concave in $\boldsymbol{x}$, $f_2(\boldsymbol{x}, \theta)$ and $g_2(\boldsymbol{x}, \theta)$ are continuous in $(\boldsymbol{x}, \theta)$, and there exists $\boldsymbol{x}_0 \in X$ such that $g(\boldsymbol{x}_0, \theta) > 0$ for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$. Then $V(\cdot)$ is absolutely continuous and hence differentiable (a.e.). In addition, for any $\boldsymbol{x}^*(\theta) \in \arg\max_{\boldsymbol{x} \in X : g(\boldsymbol{x}, \theta) \geq 0} f(\boldsymbol{x}, \theta)$,

$$V'(\theta) = \mathcal{L}_2(\boldsymbol{x}^*(\theta), \theta; \lambda^*).$$

Roy's identity and Shephard's lemma are the two canonical applications of the envelope theorem in consumer theory — one for the UMP, one for the EMP.

Corollary 2.5.4: Roy's Identity

Suppose $u(\cdot)$ represents a locally non-satiated and strictly convex preference relation on $X = \mathbb{R}^n_+$. Then, for any $(\boldsymbol{p}, m) \gg \mathbf{0}$, the Marshallian demand for good i, $x_i^M(\boldsymbol{p}, m)$, is given by:

$$x_i^M(\boldsymbol{p}, m) = -\frac{\partial v(\boldsymbol{p}, m)/\partial p_i}{\partial v(\boldsymbol{p}, m)/\partial m}.$$

Proof for Corollary.

The Lagrangian of the utility maximization problem is given by:

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu; \boldsymbol{p}, m) = u(x_1, x_2, \cdots, x_n) + \lambda\left(m - \sum_{i=1}^n p_i x_i\right) + \sum_{i=1}^n \mu_i x_i$$

By the envelope theorem, we have:

$$\begin{cases} \frac{\partial v(\boldsymbol{p},m)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}|_{(\boldsymbol{x}^*,\lambda^*,\mu^*)} = -\lambda^* x_i^M(\boldsymbol{p}, m) \\ \frac{\partial v(\boldsymbol{p},m)}{\partial m} = \frac{\partial \mathcal{L}}{\partial m}|_{(\boldsymbol{x}^*,\lambda^*,\mu^*)} = \lambda^* \end{cases} \implies x_i^M(\boldsymbol{p}, m) = -\frac{\partial v(\boldsymbol{p}, m)/\partial p_i}{\partial v(\boldsymbol{p}, m)/\partial m}$$

■

Corollary 2.5.5: Shephard's Lemma

Suppose $u(\cdot)$ represents a locally non-satiated and strictly convex preference relation on $X = \mathbb{R}^n_+$. Then, for any $\boldsymbol{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$, the Hicksian demand for good i, $x_i^H(\boldsymbol{p}, u)$, is given by:

$$x_i^H(\boldsymbol{p}, u) = \frac{\partial e(\boldsymbol{p}, u)}{\partial p_i}.$$

Proof for Corollary.

The Lagrangian of the expenditure minimization problem is given by:

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu; \boldsymbol{p}, u) = \sum_{i=1}^n p_i x_i - \lambda(u(\boldsymbol{x}) - u) - \sum_{i=1}^n \mu_i x_i.$$

By the envelope theorem, we have:

$$\frac{\partial e(\boldsymbol{p}, u)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}|_{(\boldsymbol{x}^*,\lambda^*,\mu^*)} = x_i^H(\boldsymbol{p}, u).$$

■

By strict convexity, both Marshallian demand and Hicksian demand are single-valued.

2.5.3 Slutsky Equation

The Slutsky equation is the formal decomposition we have been building toward: it splits the Marshallian response to a price change into a substitution effect (the Hicksian piece) and an income effect.

Theorem 2.5.6: Slutsky Equation

Suppose $u(\cdot)$ represents a continuous, locally non-satiated and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^n_+$ and that $\boldsymbol{x}^M(\boldsymbol{p}, m)$ and $\boldsymbol{x}^H(p, u)$ are both differentiable and single-valued. Then

$$\frac{\partial x_i^M(\boldsymbol{p}, m)}{\partial p_j} = \frac{\partial x_i^H(\boldsymbol{p}, v(\boldsymbol{p}, m))}{\partial p_j} - \frac{\partial x_i^M(\boldsymbol{p}, m)}{\partial m} x_j^M(\boldsymbol{p}, m).$$

Proof for Theorem

The proof proceeds with duality of Marshallian and Hicksian demand throughout.

$$x_i^H(\boldsymbol{p}, u) = x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, u)), \text{ as long as } u \geq u(\boldsymbol{0})$$

Take partial derivatives with respect to p_j:

$$\frac{\partial x_i^H(\boldsymbol{p}, u)}{\partial p_j} = \frac{x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, u))}{\partial p_j} + \frac{\partial x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, u))}{\partial m} \cdot \frac{\partial e(\boldsymbol{p}, u)}{\partial p_j}$$

By Shephard's lemma:

$$\frac{\partial e(\boldsymbol{p}, u)}{\partial p_j} = x_j^H(\boldsymbol{p}, u)$$

By duality,

$$\begin{cases} x_j^H(\boldsymbol{p}, v(\boldsymbol{p}, m)) = x_j^M(\boldsymbol{p}, m) \\ e(\boldsymbol{p}, v(\boldsymbol{p}, m)) = m \end{cases}$$

Evaluating the partial derivative equation at $u = v(\boldsymbol{p}, m)$, we have:

$$\frac{\partial x_i^H(\boldsymbol{p}, v(\boldsymbol{p}, m))}{\partial p_j} = \frac{x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, v(\boldsymbol{p}, m)))}{\partial p_j} + \frac{\partial x_i^M(\boldsymbol{p}, m)}{\partial m} \cdot x_j^M(\boldsymbol{p}, m)$$

Rearrange the terms to finish the proof of Slutsky Equation. ■

The economic content: the LHS is the total Marshallian response of good i's demand to a change in p_j; the first term on the RHS is the substitution effect (how the consumer would re-allocate at a fixed utility level), and the second term is the income effect (the price change effectively makes the consumer richer or poorer at the original bundle, by an amount proportional to her current consumption of good j).

The Slutsky equation seeds a battery of comparative-statics classifications. We introduce them next.

Comparative statics in income. A normal good is one for which demand rises with wealth; an inferior good is one for which demand falls.

Definition 2.5.7: Normal Good; Inferior Good

Good i is a *normal good* if $x_i^M(\boldsymbol{p}, m)$ is increasing in m, an *inferior good* if $x_i^M(\boldsymbol{p}, m)$ is decreasing in m.

"Inferior" is a technical label about how demand co-moves with income; it carries no claim about the good's quality.

Comparative statics in own price. The next pair captures how Marshallian demand responds to a change in the good's own price.

Definition 2.5.8: Regular Good; Giffen Good

Good i is a *regular good* if $x_i^M(\boldsymbol{p}, m)$ is decreasing in p_i, a *Giffen good* if $x_i^M(\boldsymbol{p}, m)$ is increasing in p_i.

A Giffen good must be inferior. The substitution effect of a price increase always pushes demand down (the Hicksian substitution matrix is negative semi-definite); for total demand to rise *with the price, the income effect must be both positive in absolute value* and *large enough to overwhelm the substitution effect — which requires the good to be inferior.*

Comparative statics in cross-price. The last pair captures how the demand for good i responds to a price change in good j.

Definition 2.5.9: Substitute; Complement

Good i is a *substitute* for good j if $x_i^H(\boldsymbol{p}, u)$ is increasing in p_j, a *complement* for good j if $x_i^H(\boldsymbol{p}, u)$ is decreasing in p_j.

Good i being a complement for good j means that, any increase in p_j would shift part of the original share of consumption onto alternative goods other than good i and good j.

Definition 2.5.10: Gross Substitute; Gross Complement

Good i is a *gross substitute* for good j if $x_i^M(\boldsymbol{p}, m)$ is increasing in p_j, a *gross complement* for good j if $x_i^M(\boldsymbol{p}, m)$ is decreasing in p_j.

2.6 Consumer Welfare

The textbook measure of consumer welfare is consumer surplus — the area between the demand curve and the price line. But it has two limitations:

- It is a partial-equilibrium tool: it does not gracefully handle multiple simultaneous price changes.
- It has no clean interpretation in terms of utility itself.

We supplement it with two utility-based welfare measures — *compensating variation* and *equivalent variation* — both expressed in dollar units of "equivalent wealth."

Definition 2.6.1: Compensating Variation; Equivalent Variation

Suppose the initial price is $\boldsymbol{p}^0$ and $u^0 = v(\boldsymbol{p}, m)$, and that the final price is $\boldsymbol{p}'$ and $u' = v(\boldsymbol{p}', m)$. Compensating variation and equivalent variation are defined as:

1. *Compensating variation*: $CV = e(\boldsymbol{p}^0, u^0) - e(\boldsymbol{p}', u^0)$.
2. *Equivalent variation*: $EV = e(\boldsymbol{p}^0, u') - e(\boldsymbol{p}', u')$.

Notice that CV and EV have the same sign, and is positive for a price drop and negative for a price increase (though the two cases are not exhaustive).

Mathematically, by duality,

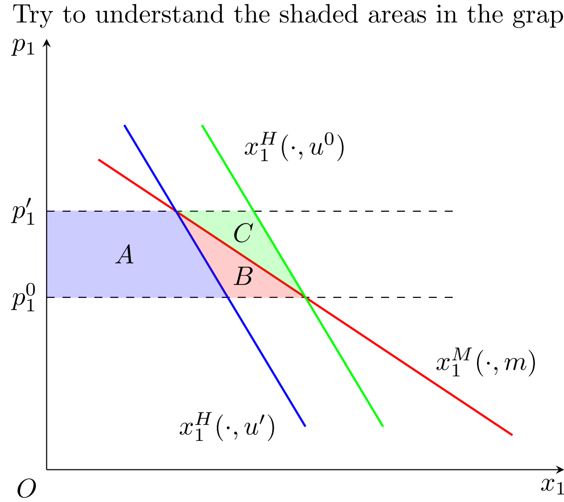
$$\begin{aligned} CV &= e(\boldsymbol{p}^0, u^0) - e(\boldsymbol{p}', u^0) = m - e(\boldsymbol{p}', u^0) \\ EV &= e(\boldsymbol{p}^0, u') - e(\boldsymbol{p}', u') = e(\boldsymbol{p}^0, u') - m \end{aligned}$$

Intuitively, $-CV$ measures how much we need to *compensate* the consumer for them to achieve the original level of utility at the new price vector, while EV measures what is the equivalent amount of money that the consumer values this price change if the price vector were fixed at the original level.

Suppose the price of a single good i changes from p_i^0 to p_i', then

$$\begin{aligned} CV &= \int_{p_i'}^{p_i^0} \frac{\partial e(\boldsymbol{p}, u^0)}{\partial p_i} \mathrm{d}p_i = \int_{p_i'}^{p_i^0} x_i^H(\boldsymbol{p}, u^0) \mathrm{d}p_i \\ EV &= \int_{p_i'}^{p_i^0} \frac{\partial e(\boldsymbol{p}, u')}{\partial p_i} \mathrm{d}p_i = \int_{p_i'}^{p_i^0} x_i^H(\boldsymbol{p}, u') \mathrm{d}p_i \\ \Delta CS &= \int_{p_i'}^{p_i^0} x_i^M(\boldsymbol{p}, m) \mathrm{d}p_i \end{aligned}$$

Suppose the price of good 1 increases from p_1^0 to p_1'. The change is depicted in the following graph, with Marshallian demand, Hicksian demands and the change of CV, EV and ΔCS. Try to understand the shaded areas in the graph.



$$\begin{cases} -CV = A + B + C \\ -EV = A \\ -\Delta CS = A + B \end{cases}$$

Remark.

1. If the Marshallian demand curve is steeper than the Hicksian demand curve, it implies that the good is an inferior good. In the graph, the represented good is a normal good.

2. On any range where the good in question is either normal or inferior, then:

$$\min\{CV, EV\} \leq \Delta CS \leq \max\{CV, EV\}.$$

Notice that the two cases are not exhaustive. For example, a good may be normal good at some lower ranger of price, but reversed to inferior good at higher range of price.

Example.

Suppose a consumer has a locally non-satiated and strictly convex preference relation on $\mathbb{R}^2_+$ that can be represented by a twice continuously differentiable utility function $u(x_1, x_2) \geq 0$. Moreover, for $(p_1, p_2) \gg \mathbf{0}$ and $u \geq 0$, the expenditure function is given by:

$$e(p_1, p_2, u) = \frac{p_1 p_2 u^2}{p_1 + p_2}$$

1. For $(p_1, p_2) \gg \mathbf{0}$ and $u > 0$, derive the Hicksian demand $\boldsymbol{x}^H(p_1, p_2, u)$.

2. For $(p_1, p_2, m) \gg \mathbf{0}$, derive the Marshallian demand $\boldsymbol{x}^M(p_1, p_2, m)$.

3. Now suppose $p_2 = 1$ and $m = 2$. Consider a price drop from $p_1^0 = 2$ to $p_1' = 1$. Calculate the compensating variation (CV), the equivalent variation (EV), and the change in consumer surplus (ΔCS) of this price change.

Solution.

1. By Shephard's Lemma,

$$x_1^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{p_2^2 u^2}{(p_1 + p_2)^2}$$
$$x_2^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{p_1^2 u^2}{(p_1 + p_2)^2}$$

It follows that $\boldsymbol{x}^H(p_1, p_2, u) = \left(\frac{p_2^2 u^2}{(p_1+p_2)^2}, \frac{p_1^2 u^2}{(p_1+p_2)^2}\right)$.

2. By duality,

$$e(p_1, p_2, v(\boldsymbol{p}, m)) = \frac{p_1 p_2 v(\boldsymbol{p}, m)^2}{p_1 + p_2} = m.$$

We then have $v(\boldsymbol{p}, m)^2 = \frac{p_1+p_2}{p_1 p_2} m$.

Again by duality,

$$\boldsymbol{x}^M(p_1, p_2, m) = \boldsymbol{x}^H(p_1, p_2, v(\boldsymbol{p}, m)) = \left(\frac{p_2 m}{p_1(p_1 + p_2)}, \frac{p_1 m}{p_2(p_1 + p_2)}\right).$$

3. Apply the definition of CV, EV and ΔCS:

$$\begin{cases} CV = e(p_1^0, p_2, u^0) - e(p_1', p_2, u') = m - e(p_1', p_2, u^0) \\ EV = e(p_1^0, p_2, u') - e(p_1^0, p_2, u') = e(p_1^0, p_2, u') - m \\ \Delta CS = \int_1^2 x_1^M(p_1, p_2, m)\mathrm{d}p_1 = 2\ln\frac{4}{3} = 0.58 \end{cases}$$

Notice that $u^0 = v(p_1^0, p_2, m) = \sqrt{3}$ and $u' = v(p_1', p_2, m) = 2$. Then the results are:

$$\begin{cases} CV = \frac{1}{2} \\ EV = \frac{2}{3} \\ \Delta CS = 2\ln\frac{4}{3} \approx 0.58 \end{cases}$$

Note that typically we have $\min\{CV, EV\} \leq \Delta CS \leq \max\{CV, EV\}$. This can be used to double check the "correctness" of your result.

There is one thing noteworthy about aggregation. While we have laid the foundation for individually analyzing consumer's utility maximization problem, it may fail if we directly make the aggregation.

Example.

Consider two consumers 1 and 2, whose preferences can be represented by the following utility functions:

$$u^1(x_1, x_2) = \begin{cases} x_1 x_2^3 & \text{if } 0 \leq x_2 \leq 7.7 \\ (7.7)^3 x_1 & \text{if } x_2 \geq 7.7 \end{cases}$$

$$u^2(x_1, x_2) = \begin{cases} x_1^3 x_2 & \text{if } x_1 \geq 3x_2 \\ \frac{1}{3}x_1^4 & \text{if } 0 \leq x_1 \leq 3x_2 \end{cases}$$

Consider the following budget sets:

- Budget set A: $p_1 = p_2 = 2$, $m = 20$.
- Budget set B: $p_1 = 3$, $p_2 = 1$, $m = 20$.

Intuitively, the failure of aggregation is due to *diverse income effects*. For instance, in the example above, the price change has a positive income effect on consumer 1, but a negative income effect on consumer 2. Aggregation is possible in the special case where all consumers have the same wealth effect, that is, $\frac{\partial \boldsymbol{x}^i}{\partial m^i} = \frac{\partial \boldsymbol{x}^j}{\partial m^j}$, for every two consumers i, j and p, m^i, m^j.

Chapter 3

Production Theory

Production theory parallels consumer theory, with firms in place of consumers and profit in place of utility. We will build the firm's problem in stages: first describe what the firm *can* do (the production set), then ask what the firm *wants* (profit maximization or, when output is given, cost minimization).

3.1 Setups

We begin with the standard assumptions that justify treating the firm as a profit-maximizer in a competitive market.

Assumption 3.1.1

1. *Perfect/complete* information: no uncertainty about input/output prices, production technology, etc.
2. *Perfectly competitive* input and output markets: firms are price-takers in both input and output markets.
3. Input/output prices are *linear*, justified by perfectly competitive markets.
4. Goods are perfectly *divisible*.
5. The technology is *exogenously given*.
6. The firm's managers are *perfectly controlled by the owners/shareholders*.

Remark.

1. Assumptions 1, 2, and 6 are what make "maximize profit" an unambiguous objective. Drop any of them and the objective itself becomes contestable:
   - Drop 1 (perfect information): owners with different risk preferences will disagree about which uncertain profit stream to maximize.
   - Drop 2 (price-taking): an owner who also has market power on the input or output side may prefer outcomes that distort the firm away from profit maximization.

- Drop 6 (perfect alignment): managers may pursue their own objectives — the agency problem.

2. Why is profit maximization determined by assumption 6 specifically? Consider a firm jointly owned by I consumers, with consumer i holding share $\theta_i \geq 0$ and $\sum_i \theta_i = 1$. Consumer i's utility maximization problem is

$$\max_{\boldsymbol{x}_i \geq \mathbf{0}} u_i(\boldsymbol{x}_i)$$
$$\text{s.t. } \boldsymbol{p} \cdot \boldsymbol{x}_i \leq m_i + \theta_i \boldsymbol{p} \cdot \boldsymbol{y}$$

Under local non-satiation, $v(\boldsymbol{p}, m)$ is strictly increasing in m. Every shareholder therefore strictly prefers higher $\boldsymbol{p} \cdot \boldsymbol{y}$ — the firm's profit — regardless of how heterogeneous their preferences over consumption bundles are. The unanimous shareholder objective is profit maximization. Drop assumption 6 (perfect manager-owner alignment) and managers might not implement this objective.

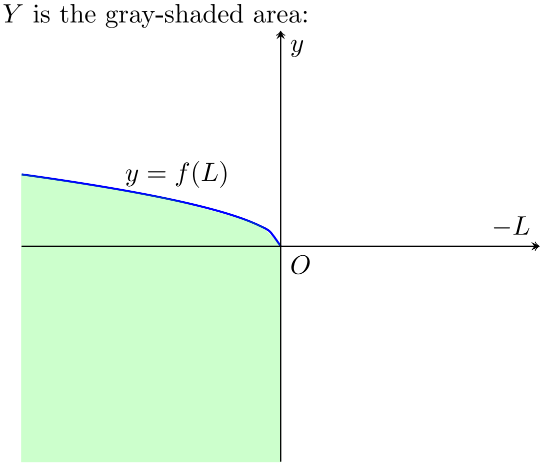
3.1.1 Production Set

Definition 3.1.2: Production Plan; Production Set

A *production plan* is a vector $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, where y_i can be either positive or negative, with $y_i > 0$ standing for an output and $y_i < 0$ an input.
The *production set* of a firm is described by $Y \subset \mathbb{R}^n$, where any $\boldsymbol{y} \in Y$ is a *feasible* production plan, i.e., a production plan the firm can choose from.

Example.

For example, consider the example of one input and one output, suppose the production set is given by $Y = \{(-L, y) : y \leq f(L), L \geq 0\}$, where $f(\cdot)$ is the production function. The production set Y is the gray-shaded area:



Throughout our analysis, we will make the innocent technical assumptions that Y is *non-empty* (so as to have something to study), *closed* (to help ensure the existence of optimal production plans), and $Y \neq \mathbb{R}^n$ (so that there is some scarcity). In addition to those, some other assumptions are needed to make the problem more practical.

Assumption 3.1.3: Production Set

1. $Y \neq \varnothing$.
2. Y is *closed*.
3. *No free lunch* and the possibility of *shutdown*: $Y \cap \mathbb{R}^n_+ = \{\mathbf{0}\}$.
4. *Free disposal*: $\boldsymbol{y} \in Y \implies \boldsymbol{y}' \in Y$, for any $\boldsymbol{y}' \leq \boldsymbol{y}$.

Remark.

Recall the distinction between the short run and the long run from intermediate microeconomics:

- Short run: some inputs are fixed.
- Long run: all inputs are variable.

In our discussion, we will mostly focus on the **long run** in advanced microeconomics. That is, all inputs are by default changeable.

3.1.2 Firm's Profit Maximization Problem

In the spirit of rational decision-making, the firm's problem can be framed as choosing the profit-maximizing production plan from its production set.

Definition 3.1.4: Profit Maximization Problem

$$\begin{aligned} &\max_{\boldsymbol{y}} \boldsymbol{p} \cdot \boldsymbol{y} \\ &\text{s.t. } \boldsymbol{y} \in Y \end{aligned}$$

Similarly, we define the firm's profit function and optimal supply correspondence, following the same logic with consumer theory.

Definition 3.1.5: Profit Function

Profit function is defined as the optimal value function of the firm's profit given $\boldsymbol{p}$:

$$\pi(\boldsymbol{p}) = \sup_{y \in Y} \boldsymbol{p} \cdot \boldsymbol{y}.$$

Definition 3.1.6: Optimal Supply Correspondence

Optimal supply correspondence is defined as the firm's optimal choice(s) given $\boldsymbol{p}$:

$$\boldsymbol{y}^*(\boldsymbol{p}) = \arg\max_{\boldsymbol{y} \in Y} \boldsymbol{p} \cdot \boldsymbol{y} = \{\boldsymbol{y} \in Y : \boldsymbol{p} \cdot \boldsymbol{y} = \pi(\boldsymbol{p})\}.$$

Remark.

- We have not made sufficient assumptions to ensure that a maximum profit is achieved (i.e., $y^*(\boldsymbol{p}) \neq \varnothing$), so the sup in profit function cannot always be replaced with the max.

  Example.

  A firm uses labor and capital to produce its sole output. The production function is given by $f(L, K) = \sqrt{LK}$. Suppose $p = 4$ and $w = r = 1$. If the firm chooses $L = K = t$, the profit is given by $\pi = 2t$, which is unbounded in t.

- $y^*(\boldsymbol{p})$, the optimal supply correspondence, is a set-valued function, which maps an element from one set, the domain of the function, to a subset of another set.

3.2 Profit Maximization and Rationalizability

The firm's analog of consumer theory's central questions is:

1. Given (some of) the firm's supply decisions $y(\boldsymbol{p})$ — but *not* the production set Y — when is $y(\cdot)$ consistent with profit maximization for some production set? (Rationalizability.)
2. When does the firm's profit maximization problem have a solution?
3. What properties do the profit function $\pi(\cdot)$ and the optimal supply correspondence $y^*(\cdot)$ inherit from the setup?
4. How do we actually solve the firm's problem?

These mirror the revealed-preference questions for consumers, but with the inference direction reversed. In consumer revealed preference we observe the feasible set and try to recover the objective; here we observe the objective (profits at various prices) and try to recover the feasible set (the production set).

In practice, we do not know a firm's production set Y, but observe some of its supply choice $y(\boldsymbol{p})$ for $\boldsymbol{p} \in \mathbb{R}^n$. Hence, we define rationalizability on empirical meanings.

Definition 3.2.1: Rationalizability

Empirical supply correspondence $y : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *rationalized* by production set Y if $y(\boldsymbol{p}) \subset y^*(\boldsymbol{p}) = \arg\max_{\boldsymbol{y} \in Y} \boldsymbol{p} \cdot \boldsymbol{y}$ for all $\boldsymbol{p} \in \mathbb{R}^n$. Empirical supply correspondence $y(\cdot)$ is *rationalizable* if it is rationalized by some production set.

Intuitively, an observed supply correspondence $y(\cdot)$ is rationalizable if we can find a production set Y such that $y(\cdot)$ is consistent with rational decision-making.

Remark.

- We may only observe some of the optimal choice(s) at each price $\boldsymbol{p}$, so in the definition it writes "$y(\boldsymbol{p}) \subset y^*(\boldsymbol{p})$".
- Practically, it is not necessary that we observe all of the firm's optimal supply decisions at all prices. We can define without loss that $y(\boldsymbol{p}_0) = \varnothing$ if the firm's supply decision

is not observed at $\boldsymbol{p}_0$.

We are naturally interested in what we can infer about the production set Y from the empirical observations if the supply choices are rationalizable. Suppose at price $\boldsymbol{p}$ the firm chooses production plan $y(\boldsymbol{p})$. Here are two plausible inferences:

1. Plan $y(\boldsymbol{p})$ must be feasible, i.e., $y(\boldsymbol{p}) \in Y$.
2. Any production plan $\boldsymbol{y}$ other than elements in $\boldsymbol{y}(\boldsymbol{p})$ must generate no more profits than elements in $y(\boldsymbol{p})$ at price $\boldsymbol{p}$. Or equivalently, any production plan $\boldsymbol{y}$ that is more profitable than $\boldsymbol{y}(\boldsymbol{p})$ at price $\boldsymbol{p}$ cannot be feasible.

We use the first idea to construct an "inner bound" on Y defined by all choices that the firm has actually made, as they must first be feasible to be chosen. We use the second idea to construct an "outer bound" on Y, which only includes plans that do not give the firm greater profits than its observed choices at any given price $\boldsymbol{p}$.

Definition 3.2.2: Inner Bound; Outer Bound

Given empirical supply correspondence $y(\cdot)$, we define the *inner bound* of the firm's production set as:

$$Y^I = \bigcup_{\boldsymbol{p} \in \mathbb{R}^n} y(\boldsymbol{p}),$$

and the *outer bound* of the firm's production set as:

$$Y^O = \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{y} \leq \boldsymbol{p} \cdot \boldsymbol{y}_{\boldsymbol{p}}, \forall \boldsymbol{p} \in \mathbb{R}^n \text{ and } \boldsymbol{y}_{\boldsymbol{p}} \in y(\boldsymbol{p})\}.$$

Intuitively, any optimal supply choice(s) must be feasible, so $Y^I \subset Y$; and any feasible production plan cannot yield a strictly higher profit at any price, so $Y \subset Y^O$. This intuition is formally characterized in the following proposition.

Proposition 3.2.3: Rationalizable Empirical Supply Correspondence

Production set Y rationalizes empirical supply correspondence $y(\cdot)$ if and only if

$$Y^I \subset Y \subset Y^O.$$

Proof for Proposition.

1. "Only if"
   - First consider any $\boldsymbol{z} \in Y^I$. By definition of Y^I, there exists a $\boldsymbol{p}$ such that $\boldsymbol{z} \in y(\boldsymbol{p})$. Since $y(\cdot)$ is rationalizable, $y(\boldsymbol{p}) \subset y^*(\boldsymbol{p}) \subset Y$. It follows that $\boldsymbol{z} \in Y$ and $Y^I \subset Y$.
   - Next consider any $\boldsymbol{y} \in Y$ and $\boldsymbol{p} \in \mathbb{R}^n$. Since $y(\cdot)$ is rationalizable, $y(\boldsymbol{p}) \subset y^*(\boldsymbol{p})$. By the definition of $y^*(\boldsymbol{p})$, for any $\boldsymbol{y}_{\boldsymbol{p}} \in y^*(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{y} \leq \boldsymbol{p} \cdot \boldsymbol{y}_{\boldsymbol{p}}$. It follows that $\boldsymbol{y} \in Y^O$ and $Y \subset Y^O$.
2. "If"
   - Fix $\boldsymbol{p} \in \mathbb{R}^n$ and consider any $\boldsymbol{y}_{\boldsymbol{p}} \in y(\boldsymbol{p})$.

- Since $y(\boldsymbol{p}) \subset Y^I \subset Y$, $\boldsymbol{y_p} \in Y$.
- Moreover, for any $\boldsymbol{y} \in Y \subset Y^O$, $\boldsymbol{p} \cdot \boldsymbol{y} \leq \boldsymbol{p} \cdot \boldsymbol{y_p}$. Consequently, $\boldsymbol{y_p} \in y^*(\boldsymbol{p})$.

Remark.

For the "if" part, by definition of Y^O, if we fix any $\boldsymbol{p} \in \mathbb{R}^n$, and take $\boldsymbol{y_p} \in y(\boldsymbol{p})$, $\boldsymbol{y_p}$ maximizes $\boldsymbol{p} \cdot \boldsymbol{y_p}$. However, with this condition we cannot simply conclude that $y(\cdot)$ is rationalizable, because $y_{\boldsymbol{p}}$ has to be in the production set Y, though it sounds trivial; or equivalently speaking, $\boldsymbol{y_p} \in Y^I$ may not fall in the outer bound Y^O.

The proposition indicates that, Y^I and Y^O carry all the information we have about the production set based on rational decision-making. The proposition immediately implies the following two corollaries:

Corollary 3.2.4: Weak Axiom of Profit Maximization (WAPM)

Let $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$. Empirical supply correspondence $y(\cdot)$ is rationalizable if and only if $Y^I \subset Y^O$, that is, $\boldsymbol{p} \cdot \boldsymbol{y_p} \geq \boldsymbol{p} \cdot \boldsymbol{y_{p'}}$, for any $\boldsymbol{p} \in P$ and $\boldsymbol{y_p} \in y(\boldsymbol{p})$, $\boldsymbol{y_{p'}} \in y(\boldsymbol{p}')$.

One simple consequence of this characterization is that, when checking rationalizability, we can restrict attention to supply functions rather than correspondences. (Simply put, compared with the preceding proposition, the production set Y is "left out" here.)

WAPM directly implies "law of supply".

Corollary 3.2.5: Law of Supply

Suppose empirical supply correspondence $y(\cdot)$ is rationalizable and let $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$. Then for any $\boldsymbol{p} \in P$ and $\boldsymbol{y_p} \in y(\boldsymbol{p})$, $\boldsymbol{y_{p'}} \in y(\boldsymbol{p}')$,

$$(\boldsymbol{p} - \boldsymbol{p}') \cdot (\boldsymbol{y_p} - \boldsymbol{y_{p'}}) \geq 0.$$

Proof for Corollary.

Since $y(\cdot)$ is rationalizable, by WAPM, we have

$$\boldsymbol{p} \cdot \boldsymbol{y_p} \geq \boldsymbol{p} \cdot \boldsymbol{y_{p'}}.$$

Switching the role of $\boldsymbol{p}$ and $\boldsymbol{p}'$, we have

$$\boldsymbol{p}' \cdot \boldsymbol{y_{p'}} \geq \boldsymbol{p}' \cdot \boldsymbol{y_p}$$

Adding the two equations above, we get the "law of supply".

In particular, if there is a single output and $y(\cdot)$ is single-valued, then

$$(p - p')(y(p) - y(p')) \geq 0$$

In other words, any rationalizable supply function must be **(weakly) upward sloping**.

Corollary 3.2.6

Let $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$. Empirical supply correspondence $y(\cdot)$ is rationalizable if and only if:

1. Any selection $\hat{y} : P \to \mathbb{R}^n$ is rationalizable.
2. For any two selections $\hat{y}$ and $\tilde{y}$ and any $\boldsymbol{p} \in P$, $\boldsymbol{p} \cdot \hat{y}(\boldsymbol{p}) = \boldsymbol{p} \cdot \tilde{y}(\boldsymbol{p})$. (Or equivalently, $\pi(\boldsymbol{p})$ is single-valued for each $\boldsymbol{p} \in P$.)

Remark.

- The first statement of this corollary is equivalent to WAPM applied to $\boldsymbol{p}' \neq \boldsymbol{p}$,
- The second statement of this corollary is equivalent to WAPM applied to $\boldsymbol{p}' = \boldsymbol{p}$.
- Thus, when given a supply correspondence, we only need to check that
  1. Each selection from it is a rationalizable supply function, and
  2. The profit function $\pi(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{y}$ is single-valued at any given $\boldsymbol{p} \in P$; or equivalently speaking, $\pi(\boldsymbol{p})$ does not depend on which selection is chosen.

  Since checking the second condition is trivial, we can focus on rationalizability of supply functions (single-valued correspondence).

Verifying rationalizability by checking all the WAPM inequalities is difficult when the set of observations is large. Fortunately, it turns out that when we have a continuum of observations, rationalizability can be verified much more easily using differential conditions. Specifically, we now suppose that we observe the firm's supply choices on an open convex set P of prices (e.g., P could be the set of all strictly positive price vectors).

Proposition 3.2.7: Rationalizability: Differentiable Case

Consider an empirical supply correspondence $y(\cdot)$ whose domain $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$ is an open convex set. Suppose that $\pi(\boldsymbol{p}) = \boldsymbol{p} \cdot y(\boldsymbol{p})$ is a differentiable function on $\boldsymbol{p} \in P$. Then $y(\cdot)$ is rationalizable if and only if:

1. (**Hotelling's Lemma**) $\nabla \pi(\boldsymbol{p}) = \boldsymbol{y}_{\boldsymbol{p}}$, for any $\boldsymbol{p} \in P$ and $\boldsymbol{y}_{\boldsymbol{p}} \in y(\boldsymbol{p})$.
2. $\pi(\cdot)$ is a convex function.

Proof for Proposition.

1. "If"
   - Fix $\boldsymbol{q} \in P$ and take any $\boldsymbol{y}_{\boldsymbol{q}} \in y(\boldsymbol{q})$. Consider the "difference function": $G(\boldsymbol{q}; \boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{y}_{\boldsymbol{q}} - \pi(\boldsymbol{p})$.
   - It suffices to show that, $G(\boldsymbol{q}; \cdot)$ is maximized at $\boldsymbol{p} = \boldsymbol{q}$.
   - Since $\pi(\cdot)$ is a convex function, then $G(\boldsymbol{q}; \cdot)$ is a concave function. Since $G(\boldsymbol{q}; \cdot)$ is differentiable in $\boldsymbol{p}$, the first-order condition is both necessary and sufficient.

- The F.O.C.: $\boldsymbol{y_q} - \nabla\pi(\boldsymbol{p})|_{\boldsymbol{p}=\boldsymbol{q}} = 0$, which is precisely the Hotelling's Lemma.

2. "Only if"

- The proof above also shows WAPM implies the Hotelling's lemma. Indeed, it is just an application of envelope formula. It remains to show $\pi(\cdot)$ is a convex function.
- By rationalizability, there exists Y such that $\pi(\boldsymbol{p}) = \sup_{\boldsymbol{y}\in Y} \boldsymbol{p}\cdot\boldsymbol{y}$.
- Take any $\boldsymbol{p}, \boldsymbol{q} \in P$ and $t \in (0,1)$. If $\pi(\boldsymbol{p}) = +\infty$ or $\pi(\boldsymbol{q}) = +\infty$, then clearly $\pi(t\boldsymbol{p} + (1-t)\boldsymbol{q}) \le t\pi(\boldsymbol{p}) + (1-t)\pi(\boldsymbol{q})$. Otherwise,

$$\begin{aligned}\pi(t\boldsymbol{p} + (1-t)\boldsymbol{q}) =& \sup_{\boldsymbol{y}\in Y}(t\boldsymbol{p} + (1-t)\boldsymbol{q})\cdot\boldsymbol{y} \\ \le& t\cdot\sup_{\boldsymbol{y}\in Y}\boldsymbol{p}\cdot\boldsymbol{y} + (1-t)\cdot\sup_{\boldsymbol{y}\in Y}\boldsymbol{q}\cdot\boldsymbol{y} \\ =& t\pi(\boldsymbol{p}) + (1-t)\pi(\boldsymbol{q})\end{aligned}$$

Proposition 3.2.8: Rationalizability: General Case

Consider an empirical supply function $y : P \to \mathbb{R}^n$, where $P \subset \mathbb{R}^n$ is a convex set. $y(\cdot)$ is rationalizable if and only if:

1. (**Producer Surplus Formula**): $\pi(\boldsymbol{p}) = \boldsymbol{p}\cdot y(\boldsymbol{p})$ satisfies, for any smooth path $\rho : [0,1] \to P$, with $\rho(0) = \boldsymbol{p}$ and $\rho(1) = \boldsymbol{p}'$,

$$\pi(\boldsymbol{p}') - \pi(\boldsymbol{p}) = \int_0^1 y(\rho(t))\cdot\rho'(t)\mathrm{d}t.$$

2. (**Law of Supply**): For $\boldsymbol{p}, \boldsymbol{p}' \in P$,

$$(\boldsymbol{p} - \boldsymbol{p}')\cdot(y(\boldsymbol{p}) - y(\boldsymbol{p}')) \ge 0.$$

Proof for Proposition.

- "Only if" part

  We have already shown the "law of supply", so it suffices to show the "producer surplus formula".

  Let $\phi(t) = \pi(\rho(t))$. Consider the "difference function":

$$\begin{aligned}\delta(\theta; t) :=& \rho(t)\cdot y(\rho(\theta)) - \pi(\rho(t)) \\ =& \rho(t)\cdot y(\rho(\theta)) - \phi(t)\end{aligned}$$

By rationalizability of $y(\cdot)$,

$$\begin{aligned}
&\delta(\theta;t) \leq 0 = \delta(\theta;\theta) \\
\Longrightarrow &\left.\frac{\partial \delta(\theta;t)}{\partial t}\right|_{t=\theta} = 0 \\
\Longleftrightarrow &\left.\frac{\partial \delta(\theta;t)}{\partial t}\right|_{t=\theta} = y(\rho(\theta)) \cdot \rho'(\theta) - \phi'(\theta) = 0 \\
\Longleftrightarrow &\phi'(\theta) = y(\rho(\theta)) \cdot \rho'(\theta) \\
\Longrightarrow &\phi(1) - \phi(0) = \int_0^1 y(\rho(t)) \cdot \rho'(t)\mathrm{d}t \\
\Longrightarrow &\pi(\boldsymbol{p}') - \pi(\boldsymbol{p}) = \int_0^1 y(\rho(t)) \cdot \rho'(t)\mathrm{d}t
\end{aligned}$$

In fact in order to derive the integral result from the derivatives, we have to prove the continuity of $\phi(\cdot)$. It can be shown that $\phi(\cdot)$ is Lipschitz continuous and hence absolutely continuous. The proof is rather technical and thus omitted here.

- "If" part

  In order to show the rationalizability of $y(\cdot)$, it suffices to show WAPM. Because the path integral is path-independent, take a straight line for math convenience:

$$\rho(t) = \boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p}).$$

We aim to prove

$$\pi(\boldsymbol{p}') - \boldsymbol{p}' \cdot y(\boldsymbol{p}) \geq 0.$$

Making tweaks to the difference in profit:

$$\begin{aligned}
\pi(\boldsymbol{p}') - \boldsymbol{p}' \cdot y(\boldsymbol{p}) &= \pi(\boldsymbol{p}') - \pi(\boldsymbol{p}) + \pi(\boldsymbol{p}) - \boldsymbol{p}' \cdot y(\boldsymbol{p}) \\
&= (\pi(\boldsymbol{p}') - \pi(\boldsymbol{p})) - (\boldsymbol{p}' - \boldsymbol{p}) \cdot y(\boldsymbol{p}) \\
&= \int_0^1 y(\rho(t)) \cdot \rho'(t)\mathrm{d}t - ((\boldsymbol{p}' - \boldsymbol{p}) \cdot y(\boldsymbol{p})) \\
&= \int_0^1 y(\boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p})) \cdot \rho'(t)\mathrm{d}t - ((\boldsymbol{p}' - \boldsymbol{p}) \cdot y(\boldsymbol{p})) \\
&= \int_0^1 (\boldsymbol{p}' - \boldsymbol{p}) \cdot [y(\boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p})) - y(\boldsymbol{p})]\,\mathrm{d}t
\end{aligned}$$

Let

$$\begin{cases} \boldsymbol{q}' = \boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p}) \\ \boldsymbol{q} = \boldsymbol{p} \end{cases}$$

One direct observation is that $\boldsymbol{q}' - \boldsymbol{q} = t(\boldsymbol{p}' - \boldsymbol{p})$. Therefore, we can simplify the preceding integral as:

$$\int_0^1 (\boldsymbol{p}' - \boldsymbol{p}) \cdot [y(\boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p})) - y(\boldsymbol{p})]\,\mathrm{d}t = \int_0^1 \frac{1}{t}(\boldsymbol{q}' - \boldsymbol{q}) \cdot (y(\boldsymbol{q}') - y(\boldsymbol{q}))\,\mathrm{d}t.$$

By law of supply, $(\boldsymbol{q}' - \boldsymbol{q}) \cdot (y(\boldsymbol{q}') - y(\boldsymbol{q})) \geq 0$. Therefore,

$$\int_0^1 \frac{1}{t}(\boldsymbol{q}' - \boldsymbol{q}) \cdot (y(\boldsymbol{q}') - y(\boldsymbol{q}))\,\mathrm{d}t \geq 0.$$

Example.

Consider a price-taking firm with m inputs and n outputs. Suppose we can observe the firm's input choices $\boldsymbol{z}$ for different input prices $\boldsymbol{w} \in \mathbb{R}_+^m$, but cannot observe its output choices or output prices (and may not even know the number of outputs n). We do know that output prices, whatever they are, do not change between the observations.

1. Suppose $m = 2$, and that at input prices $\boldsymbol{w}^1 = (1, 1)$ the firm chooses the input vector $\boldsymbol{z}^1 = (10, 15)$ and at input prices $\boldsymbol{w}^2 = (2, 3)$ it chooses the input vector $\boldsymbol{z}^2 = (13, 14)$. Is this pair of observations rationalizable (i.e., consistent with profit maximization for some production set and output prices)?

2. In general, give a necessary and sufficient condition for two input price-demand observations $\boldsymbol{z}^1, \boldsymbol{w}^1 \in \mathbb{R}_+^m$ and $\boldsymbol{z}^2, \boldsymbol{w}^2 \in \mathbb{R}^m$ to be rationalizable. Prove both necessity and sufficiency.

3. Now suppose instead that we have the following observations:

   - At prices $\boldsymbol{w}^1 = (1, 1)$, the firm chooses the input vector $\boldsymbol{z}^1 = (10, 15)$;
   - At prices $\boldsymbol{w}^2 = (2, 3)$, the firm chooses the input vector $\boldsymbol{z}^2 = (13, 13)$;
   - At prices $\boldsymbol{w}^3 = (4, 1)$, the firm chooses the input vector $\boldsymbol{z}^3 = (8, 9)$.

   Are these three observations jointly rationalizable?

Solution.

1. Suppose instead we know the output price $\boldsymbol{p}$ and output vectors $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, then by WAPM,

$$\begin{cases} (\boldsymbol{w}^1, \boldsymbol{p}) \cdot (-\boldsymbol{z}^1, \boldsymbol{y}^1) \geq (\boldsymbol{w}^1, \boldsymbol{p}) \cdot (-\boldsymbol{z}^2, y^2) \\ (\boldsymbol{w}^2, \boldsymbol{p}) \cdot (-\boldsymbol{z}^2, \boldsymbol{y}^2) \geq (\boldsymbol{w}^2, \boldsymbol{p}) \cdot (-\boldsymbol{z}^1, y^1) \end{cases}$$

$$\implies \begin{cases} \boldsymbol{p} \cdot \boldsymbol{y}^1 - 25 \geq \boldsymbol{p} \cdot \boldsymbol{y}^2 - 27 \\ \boldsymbol{p} \cdot \boldsymbol{y}^2 - 68 \geq \boldsymbol{p} \cdot \boldsymbol{y}^1 - 75 \end{cases}$$

$$\implies -2 \leq \boldsymbol{p} \cdot \boldsymbol{y}^1 - \boldsymbol{p} \cdot \boldsymbol{y}^2 \leq -3,$$

which apparently leads to a contradiction.

2. In the same way, rationalizability requires that

$$\begin{cases}(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,\boldsymbol{y}^1)\geq(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,y^2)\\(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,\boldsymbol{y}^2)\geq(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,y^1)\end{cases}$$
$$\Longrightarrow \boldsymbol{w}^1\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)\leq\boldsymbol{w}^2(\boldsymbol{z}^1-\boldsymbol{z}^2)$$
$$\Longrightarrow(\boldsymbol{w}^1-\boldsymbol{w}^2)\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq 0.$$

So $(\boldsymbol{w}^1-\boldsymbol{w}^2)\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq 0$ is a necessary condition. Then prove it is also sufficient.

Take any output price $\boldsymbol{p}$ and output vectors $\boldsymbol{y}^1,\boldsymbol{y}^2$ that satisfies $\boldsymbol{w}^1\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq \boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)\leq\boldsymbol{w}^2(\boldsymbol{z}^1-\boldsymbol{z}^2)$. It is trivial that output price $\boldsymbol{p}$ and production set $Y=\{(-z^1,\boldsymbol{y}^1),(-\boldsymbol{z}^2,\boldsymbol{y}^2)\}$ rationalize the pair of observations.

3. Again, when try to rationalize those choices, use necessary condition

$$\begin{cases}(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,\boldsymbol{y}^1)\geq(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,y^2)\\(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,\boldsymbol{y}^2)\geq(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,y^1)\end{cases}$$

for pairwise checks. Then we can get

$$\begin{aligned}-1\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)&\leq 0\\22\leq\boldsymbol{p}\cdot(\boldsymbol{y}^2-\boldsymbol{y}^3)&\leq 24\\-14\leq\boldsymbol{p}\cdot(\boldsymbol{y}^3-\boldsymbol{y}^1)&\leq -8\end{aligned}$$

Even though at first glance there is no apparent contradiction, if we try to take the sum of the first two inequalities:

$$\begin{cases}-1\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)\leq 0\\22\leq\boldsymbol{p}\cdot(\boldsymbol{y}^2-\boldsymbol{y}^3)\leq 24\end{cases}$$
$$\Longrightarrow 21\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^3)\leq 24$$

which contradicts the third equation. Thus, the choices cannot be rationalized.

3.3 Profit Maximization Problem

We now turn from rationalizability to the firm's optimization problem itself. The questions parallel those for the consumer's UMP:

1. When does the profit maximization problem have a solution?
2. What properties does the profit function $\pi(\cdot)$ inherit, and what about the optimal supply correspondence $y^*(\cdot)$?
3. How do we solve the problem explicitly?

3.3.1 Returns to Scale

Recall the earlier counter-example where the optimal supply correspondence is empty. The pathology was that the firm could keep replicating its production plan and earn ever-greater profit — the problem has no maximum. To pin down when this happens we classify production sets by how they behave under scaling.

Definition 3.3.1: Returns to Scale

The production set Y exhibits:

- *non-increasing returns to scale* if $\boldsymbol{y} \in Y \implies t\boldsymbol{y} \in Y, \forall t \in [0, 1]$.
- *non-decreasing returns to scale* if $\boldsymbol{y} \in Y \implies t\boldsymbol{y} \in Y, \forall t \in [1, +\infty)$.
- *constant returns to scale* if $\boldsymbol{y} \in Y \implies t\boldsymbol{y} \in Y, \forall t \geq 0$.

If the firm has non-decreasing returns to scale, there is no fundamental bound on its production capacity: scaling up the operation scales up profit. The result below makes this dichotomy precise — under non-decreasing returns, profits are either exactly zero or unbounded.

Proposition 3.3.2

If the production set $Y \neq \varnothing$ exhibits non-decreasing returns to scale, then for any $\boldsymbol{p} \in \mathbb{R}^n$, $\pi(\boldsymbol{p}) = 0$ or ∞.

Proof for Proposition.

First fix any $\boldsymbol{p} \in \mathbb{R}^n$. By the possibility of inaction or shutdown, $\boldsymbol{0} \in Y$, so $\pi(\boldsymbol{p}) \geq \boldsymbol{p} \cdot \boldsymbol{0} = 0$. Now suppose instead that $0 < \pi(\boldsymbol{p}) < \infty$, then there must exist $\boldsymbol{y}_0 \in Y$ such that $\boldsymbol{p} \cdot \boldsymbol{y}_0 > \pi(\boldsymbol{p}) - \varepsilon > 0$, for any $\varepsilon > 0$ small enough. By non-decreasing returns to scale, $t\boldsymbol{y} \in Y$, for any $t \geq 1$. Notice that $\boldsymbol{p} \cdot (t\boldsymbol{y}_0) = t(\boldsymbol{p} \cdot \boldsymbol{y}_0) > t\pi(\boldsymbol{p}) - t\varepsilon > \pi(\boldsymbol{p})$, for any $t > 1$ and $\varepsilon > 0$ small, which is a contradiction. It follows that $\pi(p) = 0$ or ∞. ■

3.3.2 Properties of Profit Function and Supply Correspondence

Proposition 3.3.3: Properties of Profit Function and Supply Correspondence

Suppose the production set Y is closed and satisfies the free disposal property. Let $\pi(\cdot)$ be the profit function and $y^*(\cdot)$ the associated optimal supply correspondence. Then for $\boldsymbol{p} \gg \mathbf{0}$,

- $\pi(\cdot)$ is homogeneous of degree 1.
- $\pi(\cdot)$ is a convex function.
- $y^*(\cdot)$ is homogeneous of degree 0.
- If Y is a convex set, then $y^*(\boldsymbol{p})$ is a convex set for all $\boldsymbol{p} \gg \mathbf{0}$. If Y is a strictly convex set, then $y^*(\boldsymbol{p})$ is either empty or single-valued.
- (Hotelling's Lemma) If $y^*(\boldsymbol{p})$ is single-valued, then $\pi(\cdot)$ is differentiable at $\boldsymbol{p}$ and $\nabla\pi(\boldsymbol{p}) = y^*(\boldsymbol{p})$, that is,

$$\frac{\partial \pi(\boldsymbol{p}_i)}{\partial p_i} = y_i^*(\boldsymbol{p}), \text{ for } i = 1, 2, \cdots, n.$$

3.3.3 Derivation of Profit Maximization Problem

In preceding discussions, we seldom delve into the benchmark of judging if a production plan falls within the production set, i.e., being feasible. One convenient way to represent production possibility sets is using a transformation function $T : \mathbb{R}^n \to \mathbb{R}$, where $T(y) \leq 0$ implies that y is feasible, and $T(y) \geq 0$ implies that y is infeasible. The set of boundary points $\{y \in \mathbb{R}^n : T(y) = 0\}$ is called the transformation frontier.

Definition 3.3.4: Profit Maximization Problem

Suppose $T(\cdot)$ is the transformation function defining the production set:

$$\max_{\boldsymbol{y}} \boldsymbol{p} \cdot \boldsymbol{y}$$
$$\text{s.t. } T(\boldsymbol{y}) \leq 0$$

The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{y}, \lambda) = \boldsymbol{p} \cdot \boldsymbol{y} - \lambda T(\boldsymbol{y}) = \sum_{i=1}^{n} p_i y_i - \lambda T(\boldsymbol{y}).$$

The F.O.C.s are:

$$\lambda \cdot \nabla T(\boldsymbol{y}) = \boldsymbol{p}.$$

For most discussions in the course, we focus on single-output cases:

$$\max_{z} p \cdot f(\boldsymbol{z}) - \sum_{i=1}^{m} w_i z_i$$
$$\text{s.t. } \boldsymbol{z} \geq \mathbf{0}$$

In this special case, the Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{z}, \mu_i) = p \cdot f(\boldsymbol{z}) - \sum_{i=1}^{m} w_i z_i + \sum_{i=1}^{m} \mu_i z_i.$$

The F.O.C. are:

$$\begin{cases} p \cdot \frac{\partial f(\boldsymbol{z})}{\partial z_i} - w_i + \mu_i = 0 \\ \mu_i z_i = 0, \mu_i \geq 0 \end{cases}, \text{ for all } i = 1, 2, \cdots, m.$$

Equivalently, the F.O.C. can be written as

$$p \cdot \frac{\partial f(\boldsymbol{z})}{\partial z_i} \leq w_i, \text{ with equality if } z_i > 0.$$

Interpretation of the first-order conditions is straightforward and intuitive. The LHS represents the firm's marginal benefit from using an additional unit of input i, while the RHS represents the marginal cost. The F.O.C. require that the marginal benefit cannot exceed the marginal cost, with equality if the input is ever used at the optimum.

Similarly, we should follow systematic procedures to solve the firm's profit maximization problem:

- Check returns to scale and whether the production technology is strictly convex and monotonic.
- If decreasing returns to scale, production technology being monotonic and strictly convex, apply the "tangency conditions":

$$p \cdot MP_i = w_i, \forall i$$

- Check whether the inputs are non-negative, whether the profit is non-negative, and whether the profit is bounded.

Example.

A firm uses two inputs, labor (L) and capital (K), to produce a single output (Y). The production function is given by:

$$f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{2}}$$

Suppose the input and output prices are $(w, r, p) \gg \mathbf{0}$. Solve the firm's profit maximization problem to derive the profit function $\pi(w, r, p)$ and optimal supply correspondence $y^*(w, r, p)$.

Solution.

- Step 1: The production function is Cobb-Douglas, and hence strictly convex and monotonic. Moreover, if $f(L,K) \geq y$, then $f(tL,tK) = t^{\frac{3}{4}} f(L,K) \geq t(L,K) \geq ty$, for any $0 \leq t \leq 1$, so decreasing returns to scale.

- Step 2: F.O.C. are given by

$$\begin{cases} [L] : p \cdot \frac{\partial f(L,K)}{\partial L} = w \\ [K] : p \cdot \frac{\partial f(L,K)}{\partial K} = r \end{cases} \implies \begin{cases} L^* = \frac{p^4}{64w^2r^2} \\ K^* = \frac{p^4}{32wr^3} \end{cases}$$

- Check non-negativity:

  Clearly, $L^*, K^* > 0$. It follows that

$$\begin{cases} y^*(w,r,p) = \left(-\frac{p^4}{64w^2r^2}, -\frac{p^4}{32wr^3}, \frac{p^3}{16wr^2}\right) \\ \pi(w,r,p) = \frac{p^4}{64wr^2} > 0 \end{cases}$$

3.4 Cost Minimization Problem

Although the PMP solves the firm's problem directly, it is worthwhile to detour through the cost minimization problem (CMP) for two reasons:

- The CMP is better-behaved than the PMP — much like the EMP relative to the UMP in consumer theory, existence and uniqueness conditions are milder.
- When the firm has monopoly power in the output market but is still a price-taker in inputs, the indirect approach (first minimize cost for each output level, then optimize output) is the only tractable way to solve the problem.

3.4.1 Setups and Properties

Definition 3.4.1: Cost Function; Conditional Factor Demand Correspondence

Let $Z(y) = \{\boldsymbol{z} \in \mathbb{R}^n_+ : f(\boldsymbol{z}) \geq y\}$ be the firm's feasible set. We define the optimal (minimal) value function as the *cost function*:

$$c(\boldsymbol{w}, y) = \inf_{\boldsymbol{z} \in Z(y)} \boldsymbol{w} \cdot \boldsymbol{z},$$

and the firm's optimal factor choice(s) as the *conditional factor demand correspondence*:

$$\boldsymbol{z}(\boldsymbol{w}, y) = \{\boldsymbol{z} \in Z(y) : \boldsymbol{w} \cdot \boldsymbol{z} = c(\boldsymbol{w}, y)\}.$$

Notice that min $f(\boldsymbol{x})$ is equivalent to $\max(-f(\boldsymbol{x}))$, so the cost minimization problem can be viewed as profit maximization problem on the restricted production set

$$Y_y = \{(-\boldsymbol{z}, y) : \boldsymbol{z} \in \mathbb{R}^n_+, y \leq f(\boldsymbol{z})\}.$$

From this near-equivalence between the CMP and the PMP, the differentiable-case rationalizability result transfers immediately.

Proposition 3.4.2

Consider a conditional factor demand function $\boldsymbol{z} : W \times \mathbb{R} \to \mathbb{R}^m$ for a fixed output y on an open convex set $W \subset \mathbb{R}^m_+$ such that $c(\boldsymbol{w}, y) = \boldsymbol{w} \cdot \boldsymbol{z}(\boldsymbol{w}, y)$ is differentiable in $\boldsymbol{w}$. Then $\boldsymbol{z}$ is rationalizable by some production function if and only if,

1. (**Shephard's lemma**) $\nabla_{\boldsymbol{w}} c(\boldsymbol{w}, y) = \boldsymbol{z}(\boldsymbol{w}, y)$.
2. $c(\cdot, y)$ is concave in $\boldsymbol{w}$.

We skipped rationalizability for the EMP and Hicksian demand in Ch. 2. The EMP works on the same logic as the CMP, so the analogous characterization — Shephard's lemma plus concavity in prices — holds for the expenditure function $e(\boldsymbol{p}, u)$ without modification.

Proposition 3.4.3: Properties of Cost Function and Conditional Factor Demand Correspondence

Suppose the production function $f(\cdot)$ is continuous and the production set Y satisfies the free disposal property. Then for $\boldsymbol{w} \gg \mathbf{0}$,

- **Existence of conditional factor demand**: If $Z(y) \neq \varnothing$, then the conditional factor demand correspondence $\boldsymbol{z}(\boldsymbol{w}, y) \neq \varnothing$.
- **Structure of conditional factor demand**: If the production technology is convex (i.e., the upper contour set $\{\boldsymbol{z} \geq \mathbf{0} : f(\boldsymbol{z}) \geq y\}$ is a convex set for any $y \geq 0$), then $\boldsymbol{z}(\boldsymbol{w}, y)$ is a convex set. If the production technology is strictly convex and $Z(y) \neq \varnothing$, then $\boldsymbol{z}(\boldsymbol{w}, y)$ is singleton.
- **Homogeneity**: $c(\boldsymbol{w}, y)$ is homogeneous of degree 1 in $\boldsymbol{w}$, and $\boldsymbol{z}(\boldsymbol{w}, y)$ is homogeneous of degree 0 in $\boldsymbol{w}$. **If the production function $f(\cdot)$ exhibits constant returns to scale, then $c(\boldsymbol{w}, y)$ and $\boldsymbol{z}(\boldsymbol{w}, y)$ are homogeneous of degree 1 in y.**
- **Monotonicity**: $c(\boldsymbol{w}, y)$ is non-decreasing in $\boldsymbol{w}$ and is strictly increasing in y for $y \geq 0$.
- **Binding production level**: For $y > 0$ and $Z(y) \neq \varnothing$, at any minimizer $\boldsymbol{z}^*$, $f(\boldsymbol{z}^*) = y$.
- **Convexity: If $f(\cdot)$ is a concave function, then $c(\boldsymbol{w}, \cdot)$ is a convex function of y.**
- **Shephard's lemma**: If $\boldsymbol{z}(\boldsymbol{w}, y)$ is single-valued, then $c(\boldsymbol{w}, y)$ is differentiable with respect to w_i and $\dfrac{\partial c(\boldsymbol{w}, y)}{\partial w_i} = z_i(\boldsymbol{w}, y)$.

The bolded properties have no counterpart in the EMP. The reason: in consumer theory utility is purely ordinal *— any monotone transformation of u represents the same preferences and gives the same EMP solution. In producer theory the production function f is* cardinal*: it specifies physical output, which has its own units and admits no monotone-transformation freedom. That cardinality is what lets us say, e.g., that the cost function is homogeneous of degree 1 in y under constant returns, or that it is convex in y when f is concave.*

3.4.2 Derivation of Cost Minimization Problem

Definition 3.4.4: Cost Minimization Problem

$$
\begin{aligned}
&\min_{\boldsymbol{z}} \boldsymbol{w} \cdot \boldsymbol{z} \\
\text{s.t. } & f(\boldsymbol{z}) \geq y \\
& z_i \geq 0, \forall i = 1, 2, \cdots, m
\end{aligned}
$$

Notice that CMP is almost identical to EMP in consumer theory. The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{z}; \lambda, \boldsymbol{\mu}) = \boldsymbol{w} \cdot \boldsymbol{z} - \lambda(f(\boldsymbol{z}) - y) - \sum_{i=1}^{n} \mu_i z_i.$$

The F.O.C.s are given by:

- w.r.t. z_i: $w_i - \lambda \dfrac{\partial f(\boldsymbol{z})}{\partial z_i} - \mu_i = 0$.
- Inequality constraints: $f(\boldsymbol{z}) \geq y$, $z_i \geq 0$, $\lambda \geq 0$, $\mu_i \geq 0$.
- Complementary slackness: $\lambda (f(\boldsymbol{z}) - y) = 0$, $\mu_i z_i = 0$.

For $y \geq 0$, we have binding production level (i.e., $f(\boldsymbol{z}) = y$), so the F.O.C. can be alternatively framed as

$$\lambda \frac{\partial f(z)}{\partial z_i} \leq w_i, \text{ with equality if } z_i > 0.$$

The economic intuition of $\lambda \dfrac{\partial f(z)}{\partial z_i} \leq w_i$ is that, the LHS corresponds to the marginal benefit, or shadow value of additional one unit of input z_i, and the RHS stands for the marginal cost of such investment. The F.O.C. state that at the optimum, the marginal benefit of inputs cannot exceed their marginal cost. Or more precisely, for those deployed inputs, their marginal benefit just equals marginal cost, while for inputs that are not ever invested, their marginal benefit must be no more than their marginal cost, otherwise the firm still have the room to cut down its cost, indicating the current solution has not reached the optimum.

Remark.

> Apply envelope theorem to the Lagrangian of CMP, we have
>
> $$\frac{\partial c(\boldsymbol{w}, y)}{\partial y} = \lambda^*.$$
>
> Specifically, λ^* **measures the firm's marginal cost of producing an additional unit of output** (valued by the firm instead of the market). Hence, another interpretation for the F.O.C. is that, the firm's internal valuation of additional input cannot exceed that of the market. To be specific, the marginal cost of production using factor i, $\frac{w_i}{MP_i}$, should be weakly greater than the marginal cost of producing additional one unit of product, with equality if the factor is employed at the optimum.

With the cost function, we can restate the firm's profit maximization problem in an indirect approach as:

$$\max_{y \geq 0} py - c(\boldsymbol{w}, y)$$

Clearly, the F.O.C. is given by

$$p \leq \frac{\partial c(\boldsymbol{w}, y)}{\partial y}, \text{ with equality if } y > 0.$$

If we relax the assumption of perfect competition and instead assume that the firm is a monopolist in the output market but a price-taker in the input market, then the firm no longer takes the output price as given. We can restate the firm's profit maximization problem as:

$$\max_{y \geq 0} p(y)\, y - c(\boldsymbol{w}, y)$$

The F.O.C. is given by

$$p'(y)\, y + p(y) \geq \frac{\partial c(\boldsymbol{w}, y)}{\partial y}, \text{ with equality if } y > 0.$$

The interpretation is similar to the perfectly competitive case. This indirect approach proves to be useful and attests to the power of cost minimization problem.

Chapter 4

Comparative Statics Analysis

A central question in economics is how an endogenous variable moves with an exogenous parameter. Take the firm's indirect single-product profit maximization:

$$\max_{y \geq 0} \; py - c(\boldsymbol{w}, y).$$

Holding the input price vector $\boldsymbol{w}$ fixed, how does the optimal supply y^* respond to the output price p?

More generally, given an objective $F : X \times \Theta \to \mathbb{R}$ with $X, \Theta \subset \mathbb{R}$, we ask how the maximizer

$$x^*(\theta) = \arg\max_{x \in X} F(x, \theta)$$

moves with the parameter θ. The two competing approaches we will develop are the *classical* (calculus-based, using the implicit function theorem) and the *monotone* or *lattice* approach (order-theoretic, requiring only complementarity). The latter is more robust — it works without differentiability, without concavity, and even when $x^*(\theta)$ is set-valued.

4.1 Univariate Comparative Statics

The classical approach uses the implicit function theorem. It requires two sets of assumptions:

- $F(\cdot, \cdot)$ is twice continuously differentiable.
- The maximizer $x^*(\theta)$ is unique and is characterized by the first-order condition.

Remark.

> A set of sufficient conditions for the second assumption is needed:
>
> - The choice set X is convex.
> - $F(\cdot, \theta)$ is strictly concave in x.
> - The solution is interior.

Under the two sets of assumptions, the first-order condition is:

$$F_x(x(\theta), \theta) = 0.$$

Differentiating on both sides with respect to θ, we have:

$$F_{xx}(x(\theta), \theta) \cdot x'(\theta) + F_{x\theta}(x(\theta), \theta) = 0$$
$$\Longrightarrow x'(\theta) = -\frac{F_{x\theta}(x(\theta), \theta)}{F_{xx}(x(\theta), \theta)}$$

If $F(\cdot, \theta)$ is strictly concave in x, then $F_{xx}(\cdot, \cdot) < 0$ (assuming no reflection point), so $x(\cdot)$ is strictly increasing in θ if $F_{x\theta}(\cdot, \cdot) > 0$.

The classical method's strength is that it delivers a closed-form expression for $x'(\theta)$, which is useful for quantitative work. The flip side is that the same machinery brings real drawbacks:

- *Technical:* the prerequisites (twice differentiability, strict concavity, interior solution) are strong, and the algebra can be tedious.
- *Substantive:* strict concavity in x is a particularly awkward assumption, because concavity is not invariant under monotone transformations — but the comparative-statics *direction* of $x^*(\theta)$ is. Asking for concavity to detect a direction that does not depend on concavity is overkill.

Example.

Taking $\boldsymbol{w}$ as given, the profit maximization problem is given by

$$\max_{y \geq 0} py - c(y)$$

We use the classical approach to determine whether (and when) the firm's supply curve is (weakly) upward sloping.

Notice that the choice set $Y = [0, +\infty)$ is convex, and when $c(\cdot)$ is strictly convex, the objective function is strictly concave. Assuming interiority, the first-order condition is given by:

$$p = c'(y(p))$$

Differentiating on both sides with respect to p, we get

$$y'(p) = \frac{1}{c''(y(p))}$$

Thus, $c(\cdot)$ being strictly convex is a sufficient condition for the supply curve being upward sloping. However, the requirement of $c(\cdot)$ being strictly convex is not always a reasonable assumption. Moreover, this assumption is not necessary. Recall that we have *Law of Supply*, which states that the firm's supply curve is weakly upward sloping without any other assumption.

The diagnosis: differentiability and concavity of $F(\cdot, \theta)$ are not what comparative statics fundamentally needs. The real driver is $F_{x\theta} \geq 0$ — the *complementarity* between x and

θ. The natural question is whether this cross-partial condition (suitably generalized) is by itself enough for $x^*(\theta)$ to be increasing. The remainder of this section develops a discrete, order-theoretic analogue that does not require F to be differentiable or concave.

Definition 4.1.1: Increasing Differences

Let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$. We say that $F(\cdot,\cdot)$ has *increasing differences* in (x, θ) if for any $x, x' \in X$ and $\theta, \theta' \in \Theta$ such that $x' > x$ and $\theta' > \theta$, we have

$$F(x', \theta') - F(x, \theta') \geq F(x', \theta) - F(x, \theta)$$

If the inequalities are strict for all such $x, x' \in X$ and $\theta, \theta' \in \Theta$, $F(\cdot,\cdot)$ has *strictly increasing differences* in (x, θ).

Intuitively, increasing differences says that the *marginal value* of x is larger at higher θ — x and θ are *complements*. Crucially, the condition is stated in terms of discrete differences, so it makes sense even when F is not differentiable, and it is invariant under monotone transformations of F (unlike concavity).

Proposition 4.1.2: Increasing Differences for Smooth Functions

Suppose $X = [\underline{x}, \bar{x}]$ and $\Theta = \left[\underline{\theta}, \bar{\theta}\right]$, where $X, \Theta \subset \mathbb{R}$.

1. If $F(\cdot,\cdot)$ is continuously differentiable in both x and θ, $F(\cdot,\cdot)$ has increasing differences in (x, θ) if and only if either of the two conditions holds:
   - $F_x(x, \cdot)$ is non-decreasing in θ for all x.
   - $F_\theta(\cdot, \theta)$ is non-decreasing in x for all θ.
2. If $F(\cdot,\cdot)$ is twice continuously differentiable in both x and θ, $F(\cdot,\cdot)$ has increasing differences in (x, θ) if and only if $F_{x\theta}(\cdot,\cdot) \geq 0$ for all (x, θ).

One last piece of plumbing. When $F(\cdot, \theta)$ is not strictly concave, $x^*(\theta)$ is a *set*, not a single number. We need a way to say that one set lies "below" another. Two natural orders on sets serve this purpose:

Definition 4.1.3: Comparison of Sets

For any two sets A and B, we say that:

- $A \leq B$ *in the strong set order* if for any $a \in A$ and $b \in B$, we have $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$.
- $A \leq B$ *pointwise* if for any $a \in A$ and $b \in B$, we have $a \leq b$.

Intuitively, the *strong set order* allows A and B to overlap (the overlap forms a shared region), but outside that overlap every A-element lies below every B-element. The *pointwise* order is stricter: every element of B must lie at or above *every* element of A, regardless of overlap.

Theorem 4.1.4: Univariate Topkis' Theorem

Let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$, and $x^*(\theta) = \arg\max_{x \in X} F(x, \theta)$. Then for any $\theta' > \theta$,

1. If $F(\cdot,\cdot)$ has increasing differences in (x, θ), then $x^*(\theta) \leq x^*(\theta')$ in the strong set order.
2. If $F(\cdot,\cdot)$ has strictly increasing differences in (x, θ), then $x^*(\theta) \leq x^*(\theta')$ pointwise.

Proof for Theorem

Take any $x \in x^*(\theta)$ and $x' \in x^*(\theta')$. Suppose $x > x'$, then by revealed preference,

$$F(x,\theta) \geq F(x',\theta)$$
$$F(x',\theta') \geq F(x,\theta')$$

By increasing differences,

$$F(x,\theta) - F(x',\theta) \leq F(x,\theta') - F(x',\theta')$$

Jointly we have:

$$0 \leq F(x,\theta) - F(x',\theta) \leq F(x,\theta') - F(x',\theta') \leq 0$$
$$\implies \begin{cases} F(x,\theta) = F(x',\theta) \\ F(x,\theta') = F(x',\theta') \end{cases} \implies \begin{cases} x \in x^*(\theta') \\ x' \in x^*(\theta) \end{cases}$$

If $F(\cdot,\cdot)$ has strictly increasing differences in (x, θ), then the combined inequality cannot hold, so it must be that $x^*(\theta) \leq x^*(\theta')$ pointwise. In particular, if $F(\cdot,\cdot)$ is twice continuously differentiable in both x and θ and $x^*(\theta)$ is single-valued, then $F_{x\theta}(\cdot,\cdot) > 0$ is sufficient for $x^*(\theta)$ to be (weakly) increasing in θ. ■

Example.

Again consider the firm's profit maximization problem. How would the firm's supply (correspondence) change with the output price p?

The objective function is $F(y,p) = py - c(\boldsymbol{w}, y)$. $Y = [0, +\infty)$ and $P = [0, +\infty)$ are both intervals in $\mathbb{R}$. Moreover, $F_p(y,p) = y$, which is strictly increasing in y, so $F(\cdot,\cdot)$ has strictly increasing differences. Therefore, the firm's supply increases with the output price p pointwise.

Example.

Consider a monopolist that faces a downward sloping demand curve $Q^D(p)$. Now suppose the government levies a unit tax t on the firm. How would the before-tax price p received by the firm change with the unit tax t?

The objective function is $F(p,t) = (p-t) Q^D(p) - c\left(Q^D(p)\right)$. $P = [0, +\infty)$ and $T = [0, +\infty)$ are both intervals in $\mathbb{R}$. Moreover, $F_t(p,t) = -Q^D(p)$, which is strictly increasing in p (since the demand curve is downward sloping), so $F(\cdot,\cdot)$ has strictly increasing differences in (p,t). Consequently, the firm's before-tax price p increases with t pointwise.

Notice that whether $x^*(\theta)$ increases/decreases with the parameter θ is an *ordinal* property, while (strictly) increasing differences is still a *cardinal* property. Indeed, we know from the discussion on consumer theory that $\max_x F(x,\theta)$ and $\max_x G(x,\theta)$ have the same set of maximizers if $G = \varphi \circ F$ for $\varphi(\cdot)$ strictly increasing. Nevertheless, G having increasing differences in (x,θ) does not necessarily mean F having (strictly) increasing differences in (x,θ). In other words, the requirement of (strictly) increasing differences is still too strong for monotone comparative statics. For our purpose, if we can find a positive and monotonic transformation φ such that $G = \varphi \circ F$ has increasing differences or strictly increasing differences in (x,θ), then we know $x^*(\theta)$ increases with θ in the strong set order or pointwise.

Example.

Consider the effects of an increase in the market size on monopoly quantity (and monopoly price). Each consumer in the market has an identical inverse function given by $p^D(q)$. Suppose the number of consumers N is exogenously given, and that the firm's cost function is $c(Q)$, where $Q = Nq$ is the total quantity sold (i.e., the number of consumers times per unit purchase). Discuss how the firm's cost function $c(\cdot)$ would affect the optimal *per-consumer* quantity $q^*(N)$ as the number of consumers N increases.

Solution.

The objective function is

$$F(q,N) = N \cdot p^D(q) \cdot q - c(Nq).$$

Notice that it is hard to check increasing differences of $F(\cdot,\cdot)$. (You may try it yourself and find the process blocked by some terms that need additional information to push forward the computation.) Consider $G(q,N) = \dfrac{F(q,N)}{N}$. Then if $G(\cdot,\cdot)$ is twice continuously differentiable (which indeed can be relaxed), we have

$$\begin{aligned} G(q,N) &= \frac{F(q,N)}{N} = p^D(q) \cdot q - \frac{c(Nq)}{N} \\ G_N(q,N) &= -\frac{q \cdot c'(Nq) \cdot N - c(Nq)}{N^2} \\ G_{Nq}(q,N) &= -qc''(Nq) \end{aligned}$$

If $c(\cdot)$ is concave, then $G_{Nq}(q,N) \geq 0$ and $G(q,N)$ has increasing differences in (q,N), so $q^*(\cdot)$ weakly increases with N. If $c(\cdot)$ is convex, then $G_{Nq}(q,N) \leq 0$ and $G(q,N)$ has increasing differences in $(q,-N)$, so $q^*(\cdot)$ weakly decreases with N.

4.2 Multivariate Comparative Statics

Consider a two-variable maximization problem:

$$\left(x_1^*\left(\theta\right), x_2^*\left(\theta\right)\right) = \arg \max_{(x_1, x_2) \in X \subset \mathbb{R}^2} F\left(x_1, x_2, \theta\right).$$

If F merely has increasing differences in (x_1, θ), that alone does not let us conclude that $x_1^*(\theta)$ is weakly increasing — *unless* x_2^* is independent of θ. The reason: when θ moves, it has a direct effect on x_1, but also an *indirect* effect channeled through x_2 (since x_2^* may shift, which in turn changes the optimal x_1). To sign the total effect we need a stronger structure on F that controls both channels at once.

In the univariate case the strong set order used $\min\{a, b\}$ and $\max\{a, b\}$. Extending this to vectors raises two questions:

- How should we define "min" and "max" of two vectors?
- Under that definition, will the result still lie in X?

For the first question, we introduce *meet* and *join* — the componentwise minimum and maximum, respectively, which give the greatest lower bound and the smallest upper bound of two vectors.

Definition 4.2.1: Meet; Join

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

- *meet* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x} \wedge \boldsymbol{y} = \left(\min\left\{x_1, y_1\right\}, \min\left\{x_2, y_2\right\}, \cdots, \min\left\{x_n, y_n\right\}\right).$$

- *join* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x} \vee \boldsymbol{y} = \left(\max\left\{x_1, y_1\right\}, \max\left\{x_2, y_2\right\}, \cdots, \max\left\{x_n, y_n\right\}\right).$$

where $\boldsymbol{x} \wedge \boldsymbol{y}$ is called the greatest lower bound of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{x} \vee \boldsymbol{y}$ is called the smallest upper bound of $\boldsymbol{x}$ and $\boldsymbol{y}$.

For the second question, we restrict attention to choice sets that are *closed under meet and join*. This structure is called a *sublattice*.

Definition 4.2.2: Sublattice

A set $X \subset \mathbb{R}^n$ is a *sublattice* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$, both $\boldsymbol{x} \wedge \boldsymbol{y} \in X$ and $\boldsymbol{x} \vee \boldsymbol{y} \in X$.

$\mathbb{R}^n$ *itself is a* lattice*, so any set in* $\mathbb{R}^n$ *is called a* sublattice.

Example.

- $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subset \mathbb{R}$, for $i = 1, 2, \cdots, n$.
- $X = \left\{\boldsymbol{x} \in \mathbb{R}_+^n : \boldsymbol{p} \cdot \boldsymbol{x} \leq m\right\}$, where $\boldsymbol{p} \gg \boldsymbol{0}$ is the price vector and $m > 0$ is the income. (NOT a sublattice)

- $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq g(x_1), \text{ for } g(\cdot) \text{ strictly increasing}\right\}$.

The multivariate analogue of increasing differences is *supermodularity* — a single condition that captures complementarity across all pairs of choice variables simultaneously.

Definition 4.2.3: Supermodularity

Let $F : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice. F is *supermodular* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$F(\boldsymbol{x} \wedge \boldsymbol{y}) + F(\boldsymbol{x} \vee \boldsymbol{y}) \geq F(\boldsymbol{x}) + F(\boldsymbol{y}).$$

Remark.

- Note when $X = X_1 \times X_2 \subset \mathbb{R}^2$, supermodularity is equivalent to increasing differences in (x_1, x_2).
- More generally, when $X = X_1 \times X_2 \times \cdots \times X_n \subset \mathbb{R}^n$, **supermodularity is equivalent to increasing differences in (x_i, x_j) for all pairs of $i \neq j$.**

Putting all together, we have the following multivariate Topkis's Theorem.

Proposition 4.2.4: Multivariate Topkis's Theorem

Let $F : X \times \Theta \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice and $\Theta \subset \mathbb{R}$. Consider $\boldsymbol{x}^*(\theta) = \arg\max_{\boldsymbol{x} \in X} F(\boldsymbol{x}, \theta)$. If F is supermodular, then for any $\theta' > \theta$ and $\boldsymbol{x} \in \boldsymbol{x}^*(\theta)$ and $\boldsymbol{x}' \in \boldsymbol{x}^*(\theta')$, we have

$$\begin{aligned} \boldsymbol{x} \wedge \boldsymbol{x}' &\in \boldsymbol{x}^*(\theta), \\ \boldsymbol{x} \vee \boldsymbol{x}' &\in \boldsymbol{x}^*(\theta'). \end{aligned}$$

Proof for Proposition.

Since X is a sublattice, $\boldsymbol{x} \wedge \boldsymbol{x}' \in X$ and $\boldsymbol{x} \vee \boldsymbol{x}' \in X$. By revealed preference,

$$\begin{cases} F(\boldsymbol{x}, \theta) \geq F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) \\ F(\boldsymbol{x}', \theta') \geq F(\boldsymbol{x} \vee \boldsymbol{x}', \theta') \end{cases} \implies F(\boldsymbol{x}, \theta) + F(\boldsymbol{x}, \theta') \geq F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) + F(\boldsymbol{x} \vee \boldsymbol{x}', \theta')$$

Since F is supermodular,

$$\begin{aligned} F(\boldsymbol{x}, \theta) + F(\boldsymbol{x}', \theta) &\leq F((\boldsymbol{x}, \theta) \wedge (\boldsymbol{x}', \theta')) + F((\boldsymbol{x}, \theta) \vee (\boldsymbol{x}', \theta')) \\ &= F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) + F(\boldsymbol{x} \vee \boldsymbol{x}', \theta') \end{aligned}$$

Jointly, it must be that

$$F(\boldsymbol{x}, \theta) + F(\boldsymbol{x}, \theta') = F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) + F(\boldsymbol{x} \vee \boldsymbol{x}', \theta')$$

Consequently, we have

$$\begin{aligned} F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) &= F(\boldsymbol{x}, \theta) \\ F(\boldsymbol{x} \vee \boldsymbol{x}', \theta') &= F(\boldsymbol{x}, \theta') \end{aligned}$$

which indicates that $\boldsymbol{x} \wedge \boldsymbol{x}' \in \boldsymbol{x}^*(\theta)$ and $\boldsymbol{x} \vee \boldsymbol{x}' \in \boldsymbol{x}^*(\theta')$. ∎

Example.

Suppose a firm uses two inputs (L and K) to produce a single output y, and the production function is given by

$$y = f(L, K)$$

where $f(\cdot,\cdot)$ is twice continuously differentiable.

1. Discuss how the optimal demand for capital $K^*(w, r, p)$ would be affected by an increase in the rental rate of the capital r.

2. Discuss how the optimal demand for labor $L^*(w, r, p)$ would be affected by an increase in the rental rate of the capital r.

Solution.

$$\pi(L, K, r) = pf(L, K) - wL - rK.$$

Immediately, $\pi_{Lr} = 0$ and $\pi_{Kr} = -1$, π is supermodular in $(-K, r)$, so $K^*(w, r, p)$ weakly decreases with r. Next consider the indirect path of r to L through K.

- If $f_{LK} > 0$, π is supermodular in $(-L, -K, r)$, so $L^*(w, r, p)$ weakly decreases with r.
- If $f_{LK} < 0$, π is supermodular in $(L, -K, r)$, so $L^*(w, r, p)$ weakly increases with r.

Example.

Consider the other form of market power, a market with a single buyer and competitive supply. Suppose the inverse supply curse $p^S(q) \geq 0$ is strictly increasing. Let $v(q)$ be the buyer's value for quantity q. Consequently, if there are q units of transaction in the market,

- The producer surplus is

$$S(q) = \int_0^q \left[p^S(q) - p^S(t)\right] \, \mathrm{d}t.$$

- The buyer's gain is

$$B(q) = v(q) - p^S(q)q.$$

1. Formulate both the buyer's and the social planner's optimization problems (no need to solve either of them).

2. How does the single buyer's optimal quantity q^B compare with that of the social planner q^*? Show your argument and explain intuitively.

Solution.

1. The buyer's optimization problem is

$$\max_{q \geq 0} v(q) - p^S(q)q.$$

The social planner's optimization problem is

$$\max_{q\geq 0} v(q) - \int_0^q p^S(t)\ \mathrm{d}t.$$

2. Consider the following optimization problem:

$$\max_{q\geq 0} w(q,\lambda) = B(q) + \lambda S(q)$$

This becomes the buyer's optimization problem when $\lambda = 0$, and the social planner's when $\lambda = 1$. If we could show how q monotonically change with λ, then we are done with this question. This is the univariate comparative statics problem, so we check the increasing differences of $w(q,\lambda)$:

$$\frac{\partial w(q,\lambda)}{\partial \lambda} = S(q)$$

Take any $q' \geq q \geq 0$. We have

$$\begin{aligned}
S(q') - S(q) &= \int_0^{q'} \left[p^S(q') - p^S(t)\right]\ \mathrm{d}t - \int_0^{q} \left[p^S(q) - p^S(t)\right]\ \mathrm{d}t \\
&= \int_q^{q'} \left[p^S(q') - p^S(q)\right]\ \mathrm{d}t + \int_0^{q} \left[p^S(q') - p^S(q)\right]\ \mathrm{d}t \\
&= \int_q^{q'} \left[p^S(q') - p^S(q)\right]\ \mathrm{d}t + \left[p^S(q') - p^S(q)\right] q \\
&\geq 0
\end{aligned}$$

where the last inequality holds because $p^S(\cdot)$ is upward sloping.

Thus, $w(q,\lambda)$ has increasing differences in (q,λ). From Topkis's Theorem, $q^*(\lambda) < q^*(\lambda')$.

Similar to increasing differences, supermodularity is a *cardinal* property, which is again too strong. Indeed, the weaker requirement, *quasi-supermodularity* serves our purpose.

Definition 4.2.5: Quasi-Supermodularity

Let $F : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice. F is *quasi-supermodular* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$F(\boldsymbol{x}) \geq F(\boldsymbol{x} \wedge \boldsymbol{y}) \Longrightarrow F(\boldsymbol{x} \vee \boldsymbol{y}) \geq F(\boldsymbol{y}),$$
$$F(\boldsymbol{x}) > F(\boldsymbol{x} \wedge \boldsymbol{y}) \Longrightarrow F(\boldsymbol{x} \vee \boldsymbol{y}) > F(\boldsymbol{y}).$$

Under such extension, the analysis can be further extended beyond the case of $X \subset \mathbb{R}^n$ and X forming a lattice structure.

Chapter 5

Uncertainty

Example.

Consider an investor who must decide how much of their initial wealth w to put into a risky asset. The risky asset can have any of the positive or negative rates of return r_i with probabilities p_i, $i = 1, 2, \cdots, n$. Suppose the investor is an expected utility maximizer and their utility for x amount of money for sure can be represented by a twice continuously differentiable, strictly increasing and strictly concave utility function $u(x)$. Let a^* be the investor's optimal amount of money to put in the risky asset. Give a necessary and sufficient condition for the investor to have strict incentives to invest in the risky asset, that is, $a^* > 0$ and a^* is strictly preferred to $a = 0$.

Solution.

Since the investor is an expected utility maximizer, optimization problem is given by:

$$\max_{0 \leq a \leq w} U(a) = \sum_{i=1}^{n} p_i \cdot u(w + ar_i)$$

A sufficient condition for $a^* > 0$ is $U'(0) = u'(w) \cdot \sum_{i=1}^{n} p_i r_i > 0$, that is, the expected rate of return $\sum_{i=1}^{n} p_i r_i > 0$.

Next suppose $\sum_{i=1}^{n} p_i r_i \leq 0$. Since $u(\cdot)$ is strictly concave, $U''(a) = \sum_{i=1}^{n} p_i \cdot r_i^2 \cdot u''(w + ar_i) \leq 0$, which implies $U(\cdot)$ is also strictly concave. Consequently, $U'(a) \leq U'(0) \leq 0$ for $a \geq 0$. It follows that the condition is also necessary.

The expected utility representation is convenient, but as written it looks ad hoc — why *expected* utility specifically, rather than some other functional of the probability distribution? The agenda of this chapter mirrors what we did under certainty: identify the axioms on the agent's preference relation that force an expected utility representation.

5.1 Setups

Definition 5.1.1: Simple Lottery; Compound Lottery

Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ denote a finite set of *sure* outcomes (i.e., without uncertainty).

- A *simple lottery* $\boldsymbol{p} = p_1 \circ \boldsymbol{x}_1 + p_2 \circ \boldsymbol{x}_2 + \cdots + p_n \circ \boldsymbol{x}_n$ $(p_1 + p_2 + \cdots + p_n = 1)$ is a probability distribution over a finite number of sure outcomes $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$. When there is no confusion, we will also use $\boldsymbol{p} = \{p_1, p_2, \cdots, p_n\}$ to denote such a simple lottery.
- A *compound lottery* $\sum_{j=1}^{k} \alpha_j \boldsymbol{p}^j$ $(\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1)$ is a probability distribution over a finite number of simple lotteries $\left\{\boldsymbol{p}^1, \boldsymbol{p}^2, \cdots, \boldsymbol{p}^k\right\}$.

We make two simplifying assumptions about how the decision-maker perceives lotteries:

Assumption 5.1.2

1. The probabilities are **objective**: the probabilities are regarded as objective or exogenously given facts by the decision-maker (in contrast to subjective evaluation).
2. The decision-maker is a **consequentialist**: decision-makers only care about the outcomes, instead of how lotteries are mixed.

   Under this assumption, the compound lottery $\sum_{j=1}^{k} \alpha_j \boldsymbol{p}^j$ is identical to the simple lottery it induces, i.e.,

$$\sum_{j=1}^{k} \alpha_j \boldsymbol{p}^j \iff \left(\sum_{j=1}^{k} \alpha_j \boldsymbol{p}_1^j, \sum_{j=1}^{k} \alpha_j \boldsymbol{p}_2^j, \cdots, \sum_{j=1}^{k} \alpha_j \boldsymbol{p}_n^j \right)$$

The first assumption can be relaxed.

Given the two assumptions, we can restrict our focus on the space of simple lotteries $\mathcal{P} = \Delta(X)$.

Definition 5.1.3: Space of Simple Lotteries

The space of simple lotteries, $\Delta(x)$, is defined as

$$\Delta(X) = \{(p_1, \cdots, p_n) : p_i \geq 0 \ \ \forall i, p_1 + \cdots + p_n = 1\}.$$

Consider the agent's preference relation $\succsim$ over $\mathcal{P}$. To ensure a utility representation $U : \mathcal{P} \to \mathbb{R}$, we maintain the three basic assumptions as in the certainty benchmark.

- Completeness: For any $\boldsymbol{p}^1, \boldsymbol{p}^2 \in \mathcal{P}$, either $\boldsymbol{p}^1 \succsim \boldsymbol{p}^2$ or $\boldsymbol{p}^2 \succsim \boldsymbol{p}^1$ (or both).
- Transitivity: For any $\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3 \in \mathcal{P}$, if $\boldsymbol{p}^1 \succsim \boldsymbol{p}^2$ and $\boldsymbol{p}^2 \succsim \boldsymbol{p}^3$, then $\boldsymbol{p}^1 \succsim \boldsymbol{p}^3$.

- Continuity: For any two sequences $\left(\boldsymbol{p}^i\right)_{i=1}^n, \left(\boldsymbol{q}^i\right)_{i=1}^n \in \mathcal{P}$, if $\boldsymbol{p}^i \succsim \boldsymbol{q}^i$ $(\forall i = 1, 2, \cdots, n)$ and $\lim_{n\to\infty} \boldsymbol{p}^i = \boldsymbol{p}^*$, $\lim_{n\to\infty} \boldsymbol{q}^i = \boldsymbol{q}^*$, then $\boldsymbol{p}^* \succsim \boldsymbol{q}^*$.

Remark.

> There is an alternative definition of continuity. Let $\succsim$ be a complete, transitive preference relation on $\mathcal{P}$. $\succsim$ is *continuous* if for any $\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3 \in \mathcal{P}$ such that $\boldsymbol{p}^1 \succsim \boldsymbol{p}^2 \succsim \boldsymbol{p}^3$, there exists $t \in [0, 1]$ such that $\boldsymbol{p}^2 \sim t\boldsymbol{p}^1 + (1-t)\,\boldsymbol{p}^3$.

To ensure an expected utility representation, we need an additional restriction.

Definition 5.1.4: Independence

A preference relation $\succsim$ on $\mathcal{P}$ satisfies *independence* if for any $\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3 \in \mathcal{P}$ and $t \in (0, 1)$,

$$\boldsymbol{p}^1 \succsim \boldsymbol{p}^2 \iff t\boldsymbol{p}^1 + (1-t)\,\boldsymbol{p}^3 \succsim t\boldsymbol{p}^2 + (1-t)\,\boldsymbol{p}^3$$

Independence says that mixing both lotteries with a common third lottery $\boldsymbol{p}^3$ does not flip the ranking. Note that $t\boldsymbol{p}^1 + (1-t)\boldsymbol{p}^3$ and $t\boldsymbol{p}^2 + (1-t)\boldsymbol{p}^3$ are formally *compound* lotteries; under the consequentialist assumption, we identify each with the simple lottery it induces.

Moreover, if the preference relation $\succsim$ on $\mathcal{P}$ satisfies independence, then for any $\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3 \in \mathcal{P}$ and $t \in (0, 1)$, we have

- $\boldsymbol{p}^1 \succ \boldsymbol{p}^2 \iff t\boldsymbol{p}^1 + (1-t)\,\boldsymbol{p}^3 \succ t\boldsymbol{p}^2 + (1-t)\,\boldsymbol{p}^3$
- $\boldsymbol{p}^1 \sim \boldsymbol{p}^2 \iff t\boldsymbol{p}^1 + (1-t)\,\boldsymbol{p}^3 \sim t\boldsymbol{p}^2 + (1-t)\,\boldsymbol{p}^3$
- If $\boldsymbol{p}^1 \succsim \boldsymbol{p}^2$ and $\boldsymbol{p}^3 \succsim \boldsymbol{p}^4$, then $t\boldsymbol{p}^1 + (1-t)\,\boldsymbol{p}^3 \succsim t\boldsymbol{p}^2 + (1-t)\,\boldsymbol{p}^4$.

In summary, the four assumptions we maintain throughout this chapter — completeness, transitivity, continuity, and independence — are exactly what is needed for an expected utility representation.

Assumption 5.1.5

- **Completeness**: For any $\boldsymbol{p}^1, \boldsymbol{p}^2 \in \mathcal{P}$, either $\boldsymbol{p}^1 \succsim \boldsymbol{p}^2$ or $\boldsymbol{p}^2 \succsim \boldsymbol{p}^1$ (or both).
- **Transitivity**: For any $\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3 \in \mathcal{P}$, if $\boldsymbol{p}^1 \succsim \boldsymbol{p}^2$ and $\boldsymbol{p}^2 \succsim \boldsymbol{p}^3$, then $\boldsymbol{p}^1 \succsim \boldsymbol{p}^3$.
- **Continuity**: For any two sequences $\left(\boldsymbol{p}^i\right)_{i=1}^n, \left(\boldsymbol{q}^i\right)_{i=1}^n \in \mathcal{P}$, if $\boldsymbol{p}^i \succsim \boldsymbol{q}^i$ for all i and $\lim_{n\to\infty} \boldsymbol{p}^i = \boldsymbol{p}^*$, $\lim_{n\to\infty} \boldsymbol{q}^i = \boldsymbol{q}^*$, then $\boldsymbol{p}^* \succsim \boldsymbol{q}^*$.
- **Independence**: For any $\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3 \in \mathcal{P}$ and any $t \in (0, 1)$,

$$\boldsymbol{p}^1 \succsim \boldsymbol{p}^2 \iff t\boldsymbol{p}^1 + (1-t)\boldsymbol{p}^3 \succsim t\boldsymbol{p}^2 + (1-t)\boldsymbol{p}^3.$$

5.2 Expected Utility Representation

Before starting the representation theorem, we first formalize what we mean by an expected utility function.

Definition 5.2.1: Expected Utility Form

A utility function $U : \mathcal{P} \to \mathbb{R}$ has an *expected utility form* (or a *von Neumann-Morgenstern expected utility function*) if there is an assignment of numbers $(u_1, u_2, \cdots, u_n)$ to each of the n sure outcomes $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ such that for any $\boldsymbol{p} \in \mathcal{P}$,

$$U(\boldsymbol{p}) = \sum_{i=1}^{n} p_i u_i.$$

A sure outcome $\boldsymbol{x}_i$ can be represented by a simple lottery $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ and we have $U(\mathbf{e}_i) = u(\boldsymbol{x}_i) = u_i$. Notice that an expected utility function can be written as

$$U(\boldsymbol{p}) = \sum_{i=1}^{n} p_i U(\mathbf{e}_i).$$

Thus, $U(\boldsymbol{p})$ is linear in the probabilities. The following proposition reveals that this observation rings true more generally.

Proposition 5.2.2

A linear function $U : \mathcal{P} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, for any k (simple) lotteries $\boldsymbol{p}^j \in \mathcal{P}, (j = 1, 2, \cdots, k)$ and associated probabilities $\alpha_j \geq 0$ with $\sum_{j=1}^{k} \alpha_j = 1$, we have

$$U\left(\sum_{j=1}^{k} \alpha_j \boldsymbol{p}^j\right) = \sum_{j=1}^{k} \alpha_j U(\boldsymbol{p}^j).$$

Proof for Proposition.

- "If" part

  Let $\boldsymbol{p}^j = \boldsymbol{e}_j$ for all $j = 1, 2, \cdots, k$. Thus, $\boldsymbol{p} = \sum_{j=1}^{k} \alpha_j \boldsymbol{p}^j = (\alpha_1, \alpha_2, \cdots, \alpha_k)$. The condition of U being linear is then transformed to

  $$U(\boldsymbol{p}) = \sum_{j=1}^{k} p_j U(\mathbf{e}_j) = \sum_{j=1}^{k} \alpha_j u_j.$$

  This directly implies that U has an expected utility form.

- "Only if" part

  – Since the decision-maker is a consequentialist, the compound lottery $\sum_{j=1}^{k} \alpha_j \circ \boldsymbol{p}^j$

is identical to the simple lottery it induces,

$$\alpha_1 \circ \boldsymbol{p}^1 + \alpha_2 \circ \boldsymbol{p}^2 + \cdots + \alpha_k \circ \boldsymbol{p}^k \iff \left(\sum_{j=1}^{k} \alpha_j p_1^j, \sum_{j=1}^{k} \alpha_j p_2^j, \cdots, \sum_{j=1}^{k} \alpha_j p_n^j \right)$$

– Since the utility function has the expected utility form, we have

$$U\left(\sum_{j=1}^{k} \alpha_j p_1^j, \sum_{j=1}^{k} \alpha_j p_2^j, \cdots, \sum_{j=1}^{k} \alpha_j p_n^j \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{k} \alpha_j p_i^j \right) u_i$$

– On the other hand, $U\left(\boldsymbol{p}^j\right) = \sum_{i=1}^{n} p_i u_i$, so

$$\sum_{j=1}^{k} \alpha_j U\left(\boldsymbol{p}^j\right) = \sum_{j=1}^{k} \alpha_j \left(\sum_{i=1}^{n} p_i^j u_i \right)$$

– Jointly, we have proved the result:

$$\begin{aligned} U\left(\sum_{j=1}^{k} \alpha_j \boldsymbol{p}^j \right) &= U\left(\sum_{j=1}^{k} \alpha_j p_1^j, \sum_{j=1}^{k} \alpha_j p_2^j, \cdots, \sum_{j=1}^{k} \alpha_j p_n^j \right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{k} \alpha_j p_i^j \right) u_i \\ &= \sum_{j=1}^{k} \alpha_j U\left(\boldsymbol{p}^j\right) \end{aligned}$$

Theorem 5.2.3

A complete and transitive preference relation $\succsim$ on $\mathcal{P}$ satisfies continuity and independence if and only if it admits an expected utility representation $U : \mathcal{P} \to \mathbb{R}$.

Proof for Theorem

The "if" part is easy to verify. We mainly focus on the "only if" part.

Suppose the complete and transitive preference relation $\succsim$ satisfies continuity and independence. We construct an expected utility function $U(\boldsymbol{p}) = \sum_{i=1}^{n} p_i u_i$ that represents $\succsim$ as follows:

- First, with a slight abuse of notation, interpret the consequences $x_1, \cdots, x_n$ as degenerate lotteries, so that x_i puts probability 1 on x_i and probability 0 on all other consequences. Then any lottery p can be viewed as a mixture of those degenerate lotteries: $p = \sum_i p_i x_i$.

- Label the consequences so that (when viewed as degenerate lotteries) x_1 is the best and x_n is the worst, i.e., $x_1 \succsim x_i \succsim x_n$ for all i.

- By continuity, for each consequence x_i, there is some $\lambda_i \in [0, 1]$ such that $x_i \sim \lambda_i x_1 +$

$(1-\lambda_i)\, x_n$.

- By independence (iterated n times for each consequence), for any lottery $p = \sum_i p_i x_i$,

$$p \sim \sum_i p_i \left(\lambda_i x_1 + (1-\lambda_i)\, x_n\right) = \left(\sum_i p_i \lambda_i\right) x_1 + \left(1 - \sum_i p_i \lambda_i\right) x_n$$

  – If $x_1 \sim x_n$ then by independence, for all p we have

$$p \sim \left(\sum_i p_i \lambda_i\right) x_1 + \left(1 - \sum_i p_i \lambda_i\right) x_1 = x_1$$

  Hence, letting $u_i = 0$ for all i gives an expected utility representation of $\succsim$.

  – If $x_1 \succ x_n$, then for $1 \geq \lambda \geq \lambda' \geq 0$, letting $\delta = \lambda - \lambda' \in [0,1]$ and $\hat{p} = \frac{\lambda'}{1-\delta} x_1 + \frac{1-\lambda}{1-\delta} x_n$ and using the Independence Axiom,

$$\begin{aligned} \lambda x_1 + (1-\lambda)\, x_n &= \delta x_1 + (1-\delta)\, \hat{p} \\ &\succ \delta x_1 + (1-\delta)\, \hat{p} \\ &= \lambda' x_n + (1-\lambda')\, x_n \end{aligned}$$

  Thus, letting $u_i = \lambda_i$ yields an expected utility representation of $\succsim$.

5.3 Measures of Risk

5.3.1 Risk Attitudes

Let the set of sure outcomes $X = \mathbb{R}$. For any sure outcome $x \in X$, its utility is determined by utility function $u : X \to \mathbb{R}$. A lottery is fully characterized by a corresponding distribution function F, whose expected utility is then $U(F) = \int_X u(x)\, \mathrm{d}F(x)$. For simplicity, suppose $u(\cdot)$ is strictly increasing and continuous and $U(F) = \mathbb{E}_F[u(x)] < +\infty$.

Here we naturally extend the notion of lottery to $X = \mathbb{R}$, the infinite set, and avoid any technical issues related to the definition of lotteries on finite-many sure outcomes.

Definition 5.3.1: Risk Averse

For any non-degenerate lottery F, define its average payoff δ_{E_F} as

$$\delta_{\mathrm{E}_F} = \int_X x \mathrm{d}F(x)\,.$$

A decision-maker is strictly *risk-averse* if for any non-degenerate lottery F, the sure outcome δ_{E_F} is strictly preferred to the lottery F, i.e., $\delta_{\mathrm{E}_F} \succ F$. Or in utility terms,

$$u(\delta_{\mathrm{E}_F}) = u\left(\int_X x \mathrm{d}F(x)\right) > U(F) = \int_X u(x)\, \mathrm{d}F(x)$$

Corollary 5.3.2

A decision-maker is (strictly) risk-averse if and only if $u(\cdot)$ is (strictly) concave.

Intuitively, a risk-averse individual has *decreasing* marginal utility — each additional dollar is worth less than the previous one. *Risk-loving* (u convex, $F \succ \delta_{\mathbb{E}[F]}$) and *risk-neutral* ($u$ linear, $F \sim \delta_{\mathbb{E}[F]}$) attitudes are defined symmetrically.

Risk attitudes contrast the utility of an average payoff with the average utility of the payoffs. The *certainty equivalent* quantifies how large the gap between the two is — in monetary units.

Definition 5.3.3: Certainty Equivalent

The *certainty equivalent* $c(F,u)$ of a money lottery is defined as

$$u(c(F,u)) = U(F).$$

Intuitively, $c(F,u)$ is the sure amount of money the agent considers equivalent to facing the risky lottery F.

Certainty equivalent depends on initial wealth. For simplicity, we assume $w_0 = 0$, i.e., the agent does not have initial wealth; then $c(F,u) \leq \mathbb{E}[F]$ if and only if the agent is risk-averse.

5.3.2 Measures of Risk Aversion

So far "risk-averse" is a yes/no property, controlled by concavity of u. The vNM expected utility is cardinally meaningful (only up to positive affine transformations), which raises the natural follow-up question: can we use u to *quantify* how risk-averse an agent is, and rank two agents accordingly? The two standard measures below do exactly that.

Consider a risk-averse agent who has an initial wealth w and faces a (small) fair gamble.

- Scenario 1: The gamble $\tilde{\varepsilon}$ is measured in terms of monetary unit. Then consider how large a has to be to make
$$u(w-a) = U(w+\tilde{\varepsilon}).$$
- Scenario 2: The gamble $\tilde{\delta}$ is measured in terms of percentage of the initial wealth. Then consider how large r has to be to ensure
$$u((1-r)w) = U\left(\left(1+\tilde{\delta}\right)w\right).$$

Since the agent is risk-averse, $a, r > 0$. Intuitively, a measures how much money the agent is willing to give up to avoid the gamble; r measures the percentage of initial wealth the agent is willing to give up to avoid the gamble. If the agent were risk-neutral, $a = r = 0$. As a result, given initial wealth w and the small fair gamble, both a and r measure the agent's level of risk aversion.

Coefficient of Absolute Risk Aversion When $\tilde{\varepsilon}$ is small, a is small. By Taylor expansion and definition of expected utility:

$$\begin{aligned} u(w-a) &\approx u(w) - u'(w) \cdot a \\ U(w+\tilde{\varepsilon}) &= \mathbb{E}_{\varepsilon}\left[u(w+\varepsilon)\right] = \int_{\varepsilon} u(w+\varepsilon)\,\mathrm{d}\varepsilon \\ &\approx \int_{\varepsilon}\left[u(w) + u'(w)\cdot\varepsilon + \frac{1}{2}u''(w)\cdot\varepsilon^2\right]\mathrm{d}\varepsilon \\ &= u(w)\int_{\varepsilon} 1\mathrm{d}\varepsilon + u'(w)\int_{\varepsilon}\varepsilon\mathrm{d}\varepsilon + \frac{1}{2}u''(w)\int_{\varepsilon}\varepsilon^2\mathrm{d}\varepsilon \\ &= u(w) + \frac{1}{2}u''(w)\cdot\mathrm{Var}\left[\tilde{\varepsilon}\right] \end{aligned}$$

It follows that

$$\begin{aligned} u(w) - u'(w)\cdot a &= u(w) + \frac{1}{2}u''(w)\cdot\mathrm{Var}\left[\tilde{\varepsilon}\right] \\ \Longrightarrow a &\approx -\frac{1}{2}\cdot\frac{u''(w)}{u'(w)}\cdot\mathrm{Var}\left[\tilde{\varepsilon}\right] \end{aligned}$$

Definition 5.3.4: Coefficient of Absolute Risk Aversion

Suppose the agent has initial wealth $x > 0$, and the utility function $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Then the *coefficient of absolute risk aversion*, $A(x, u)$, is defined as

$$A(x,u) = -\frac{u''(x)}{u'(x)}.$$

Coefficient of Relative Risk Aversion Similarly, we use Taylor expansion and definition of expected utility to compute r:

$$\begin{aligned} u(w(1-r)) &\approx u(w) - u'(w)\cdot(wr) \\ U\left(w + w\tilde{\delta}\right) &= \mathbb{E}_G\left[u(w+w\delta)\right] = \int_{\delta} u(w+w\delta)\,\mathrm{d}G(\delta) \\ &\approx \int\left[u(w) + u'(w)\cdot(w\delta) + \frac{1}{2}u''(w)\cdot(w\delta)^2\right]\mathrm{d}G(\delta) \\ &= u(w)\int 1\mathrm{d}G(\delta) + u'(w)\int_{\delta} w\delta\mathrm{d}G(\delta) + \frac{1}{2}u''(w)\int_{\delta}(w\delta)^2\,\mathrm{d}G(\delta) \\ &= u(w) + \frac{1}{2}u''(w)\cdot w^2\mathrm{Var}\left[\tilde{\delta}\right] \end{aligned}$$

It follows that

$$\begin{aligned} u((1-r)w) &= U\left(\left(1+\tilde{\delta}\right)w\right) \\ \Longrightarrow r &\approx -\frac{wu''(w)}{2u'(w)}\mathrm{Var}\left[\tilde{\delta}\right] \end{aligned}$$

Definition 5.3.5: Coefficient of Relative Risk Aversion

Suppose the agent has initial wealth $x > 0$, and the utility function $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Then the *coefficient of relative risk aversion*, $R(x,u)$, is defined as

$$A(x,u) = -\frac{xu''(x)}{u'(x)}.$$

Proposition 5.3.6: Equivalence of Different Measures of Risk Aversion

Suppose utility functions u and v are strictly increasing and twice differentiable. The following definitions of an agent characterized by u being more risk averse than another agent characterized by v are equivalent:

1. For any lottery F and sure outcome δ_X. if $F \succsim_u \delta_X$, then $F \succsim_v \delta_X$.
2. For any lottery F, $c(F,u) \leq c(F,v)$.
3. The function u is "more concave" than v, that is, there exists some increasing and concave function g such that $u = g \circ v$.
4. $r(x) = \dfrac{u'(x)}{v'(x)}$ is non-increasing in x.
5. For any x, $A(x,u) = -\dfrac{u''(x)}{u'(x)} \geq -\dfrac{v''(x)}{v'(x)} = A(x,v)$.

Proof for Proposition.

- (1) $\iff$ (2)
  - (1) $\implies$ (2): By definition of certainty equivalent, $\delta_{c(F,u)} \sim_u F$. By condition (1), we have $\delta_{c(F,v)} \sim_v F \succsim_v \delta_{c(F,u)}$. Recall that $v(\cdot)$ is strictly increasing, so we have $c(F,u) \leq c(F,v)$.
  - (2) $\implies$ (1): If $\delta_{c(F,u)} \sim F \succsim_u \delta_x$, then by strictly-increasing property of u, it follows that $c(F,u) \geq x$. So $c(F,v) \geq c(F,u) \geq x \implies c(F,v) \geq x$. Again by v being strictly increasing, we have $c(F,v) \sim F \succsim_v \delta_x$.
- (2) $\iff$ (3)
  - Since u and v are strictly increasing, v^{-1} is well defined and $g = u \circ v^{-1}$ is strictly increasing.
  - Again by strict monotonicity of u, we have $u(c(F,u)) \leq u(c(F,v))$ for any lottery F.
  $$u(c(F,u)) = U(F) = \int_X u(x)\,\mathrm{d}F(x) = \int_X g(v(x))\,\mathrm{d}F(x)$$
  $$u(c(F,v)) = g(v(c(F,v))) = g\left(\int_X v(x)\,\mathrm{d}F(x)\right)$$
  - By Jensen's inequality, $\int_X g(v(x))\,\mathrm{d}F(x) \leq g\left(\int_X v(x)\,\mathrm{d}F(x)\right)$ if and only if g is concave.

- (3) $\iff$ (4)
  - Since u and v are strictly increasing, v^{-1} is well defined and $g = u \circ v^{-1}$ is strictly increasing.
  - When u and v are both differentiable, $u'(x) = g'(v(x)) \cdot v'(x) \Longrightarrow \frac{u'(x)}{v'(x)} = g'(v(x))$.
  - Since v is strictly increasing, $\frac{u'(x)}{v'(x)}$ is non-increasing in x if and only if g' is non-increasing, that is, g is concave.
- (4) $\iff$ (5)
  - When $r(x) > 0$, $r(x)$ is non-increasing in x if and only if $\ln r(x) = \ln u'(x) - \ln v'(x)$ is non-increasing in x.
  - When u and v are twice differentiable,

$$(\ln r(x))' = \frac{u''(x)}{u'(x)} - \frac{v''(x)}{v'(x)}$$

  - It follows that $r(x) = \frac{u'(x)}{v'(x)}$ is non-increasing in x if and only if $A(x,u) \geq A(x,v)$.

Example.

Consider an investor who must decide how much of their initial wealth w to put into a risky asset. The risky asset can have any of the positive or negative rates of return r_i with probabilities p_i, $i = 1, 2, \cdots, n$. Suppose the investor is an expected utility maximizer and their utility for x amount of money for sure can be represented by a twice continuously differentiable, strictly increasing and strictly concave utility function $u(x)$. Let a^* be the investor's optimal amount of money to put in the risky asset. Suppose $\sum_{i=1}^{n} p_i r_i > 0$.

1. Give a sufficient condition for $a^* < w$.
2. Suppose $a^* < w$ and that $A(x,u)$ is strictly decreasing in x, how would $a^*(w)$ change with w?

5.4 Comparison of Risky Prospects

The coefficient of absolute risk aversion $A(x,u)$ lets us compare agents (or the same agent at different wealth levels). The natural next question is about comparing *lotteries*: when can we say $F \succsim G$ for *every* (risk-averse) decision-maker, regardless of the specific shape of u?

A monetary lottery is summarized by two features that any agent will care about: its expected payoff and its dispersion around that expectation. This suggests two unambiguous cases in which F should be ranked above G:

1. F gives a (weakly) higher payoff than G at every probability level — F "shifts the distribution to the right."

2. F and G have the same expected payoff, but F is less dispersed.

These two cases correspond to *first-order* and *second-order* stochastic dominance respectively.

5.4.1 First-Order Stochastic Dominance

Proposition 5.4.1: First-Order Stochastic Dominance

The following two conditions are equivalent:

1. For any non-decreasing $u : \mathbb{R} \to \mathbb{R}$,
$$\int_X u(t)\,\mathrm{d}F(t) \geq \int_X u(t)\,\mathrm{d}G(t)\,.$$
2. $F(x) \leq G(x)$ for almost every x.

In this case, F *first-order stochastically dominates* G.

Proof for Proposition.

Assume that u is continuously differentiable, and $u'(\cdot) > 0$. Let $X = [\underline{x}, \bar{x}]$.

$$\begin{aligned}
&\int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}F(t) - \int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}G(t) \\
&= \left[u(t)\,F(t)\Big|_{\underline{x}}^{\bar{x}} - \int_{\underline{x}}^{\bar{x}} u'(t)\,F(t)\,\mathrm{d}t \right] - \left[u(t)\,G(t)\Big|_{\underline{x}}^{\bar{x}} - \int_{\underline{x}}^{\bar{x}} u'(t)\,G(t)\,\mathrm{d}t \right] \\
&= \left[(u(\underline{x}) - u(\bar{x})) - \int_{\underline{x}}^{\bar{x}} u'(t)\,F(t)\,\mathrm{d}t \right] - \left[(u(\underline{x}) - u(\bar{x})) - \int_{\underline{x}}^{\bar{x}} u'(t)\,G(t)\,\mathrm{d}t \right] \\
&= \int_{\underline{x}}^{\bar{x}} u'(t)\,(G(t) - F(t))\,\mathrm{d}t
\end{aligned}$$

It follows that $\int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}F(t) \geq \int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}G(t)$ if and only if $G(x) \geq F(x)$ almost everywhere. ∎

The first condition is the universal characterization: every decision-maker who prefers more money to less ranks F above G, regardless of their risk attitude or the specific shape of u.

5.4.2 Second-Order Stochastic Dominance

Definition 5.4.2: Mean-Preserving Spread

Let X, Y be two random variables with distribution functions F and G, respectively. Then G is a *mean-preserving spread* of F if $Y = X + \tilde{\varepsilon}$, where $\mathbb{E}[\tilde{\varepsilon}|X] = 0$.

A mean-preserving spread keeps the expected payoff fixed but adds noise to the realization — G is obtained from F by spreading out the probability mass without shifting its mean.

Proposition 5.4.3: Second-Order Stochastic Dominance

Suppose $\mathbb{E}[F] = \mathbb{E}[G]$. Then the following three conditions are equivalent:

1. For any (non-decreasing) concave $u : \mathbb{R} \to \mathbb{R}$,

$$\int_X u(t)\,\mathrm{d}F(t) \geq \int_X u(t)\,\mathrm{d}G(t)$$

2. For almost every $x \in X$,

$$\int_{-\infty}^{x} F(t)\,\mathrm{d}t \leq \int_{-\infty}^{x} G(t)\,\mathrm{d}t$$

3. G is a mean-preserving spread of F.

In this case, F *second-order stochastically dominates* G.

Proof for Proposition.

For simplicity, suppose $X = [\underline{x}, \bar{x}]$ and u is twice continuously differentiable.

- (1) $\iff$ (2)
  - By earlier calculation, $\int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}F(t) - \int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}G(t) = \int_{\underline{x}}^{\bar{x}} u'(t)\,(G(t) - F(t))\,\mathrm{d}t$.
  - Again by integration by parts and noticing $\mathbb{E}[F] = \mathbb{E}[G]$, we have:

$$\int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}F(t) - \int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}G(t) = \int_{\underline{x}}^{\bar{x}} u''(x)\left[\int_{\underline{x}}^{x} G(t)\,\mathrm{d}t - \int_{\underline{x}}^{x} F(t)\,\mathrm{d}t\right]\mathrm{d}x$$

  - Because u is concave, $u'' \leq 0$. It follows that $\int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}F(t) \geq \int_{\underline{x}}^{\bar{x}} u(t)\,\mathrm{d}G(t)$ if and only if $\int_{\underline{x}}^{x} F(t)\,\mathrm{d}t \geq \int_{\underline{x}}^{x} G(t)\,\mathrm{d}t$ almost everywhere.
- (1) $\iff$ (3)
  - We will only show (3) $\implies$ (1).
  - By the law of iterated expectation,

$$\mathbb{E}_G[u(Y)] = \mathbb{E}_F[\mathbb{E}_G[u(Y)\,|X]]$$

  - Since $u(\cdot)$ is concave, by Jensen's inequality,

$$\mathbb{E}_G[u(Y)] = \mathbb{E}_F[\mathbb{E}_G[u(Y)\,|X]] \leq \mathbb{E}_F[u(\mathbb{E}_G[Y|X])] = \mathbb{E}_F[u(X)]$$

∎

The first condition says that any risk-averse decision-maker would prefer lottery F over G. Additionally, when $\mathbb{E}[F] = \mathbb{E}[G]$, we actually do not need $u(\cdot)$ to be non-decreasing. If $\mathbb{E}[F] \neq \mathbb{E}[G]$, then the second and the third conditions are still equivalent, but we actually need $u(\cdot)$ to be non-decreasing and concave.

5.4.3 Likelihood Ratio Dominance

Definition 5.4.4: Likelihood Ratio Dominance

Let F and G be two distribution functions with common support $[\underline{x}, \bar{x}]$. Suppose the density functions exist and are given by f and g, respectively. F *dominates* G *in the likelihood ratio order* if $\frac{f(x)}{g(x)}$ is non-decreasing in x.

Intuitively, when F dominates G in the likelihood ratio order, F puts higher probabilities on higher returns compared with G.

Proposition 5.4.5: Likelihood Ratio Dominance Implies First-Order Stochastic Dominance

If F dominates G in the likelihood ratio order, then F first-order stochastically dominates G.

Proof for Proposition.

Since $F(\cdot)$ and $G(\cdot)$ are both non-decreasing, in order to show that $F(x) \leq G(x)$, it suffices to show that

$$\frac{F(x)}{1-F(x)} \leq \frac{G(x)}{1-G(x)}, \forall x \in (\underline{x}, \bar{x}).$$

We have

$$\begin{aligned}\frac{F(x)}{1-F(x)} &= \frac{\int_{\underline{x}}^{x} f(t)\,\mathrm{d}t}{\int_{x}^{\bar{x}} f(t)\,\mathrm{d}t} \\ &= \frac{\int_{\underline{x}}^{x} \frac{f(t)}{g(t)} \cdot g(t)\,\mathrm{d}t}{\int_{x}^{\bar{x}} \frac{f(t)}{g(t)} \cdot g(t)\,\mathrm{d}t}\end{aligned}$$

Since $\frac{f(x)}{g(x)}$ is non-decreasing,

$$\begin{aligned}\forall t \in [\underline{x}, x], \frac{f(t)}{g(t)} &\leq \frac{f(x)}{g(x)} \\ \forall t \in [x, \bar{x}], \frac{f(t)}{g(t)} &\geq \frac{f(x)}{g(x)}\end{aligned}$$

Therefore, we have

$$\frac{F(x)}{1-F(x)} = \frac{\int_{\underline{x}}^{x} \frac{f(t)}{g(t)} \cdot g(t)\,\mathrm{d}t}{\int_{x}^{\bar{x}} \frac{f(t)}{g(t)} \cdot g(t)\,\mathrm{d}t} \leq \frac{\frac{f(x)}{g(x)} \cdot \int_{\underline{x}}^{x} g(t)\,\mathrm{d}t}{\frac{f(x)}{g(x)} \cdot \int_{x}^{\bar{x}} g(t)\,\mathrm{d}t} = \frac{\int_{\underline{x}}^{x} g(t)\,\mathrm{d}t}{\int_{x}^{\bar{x}} g(t)\,\mathrm{d}t} = \frac{G(x)}{1-G(x)}$$

5.5 Comparative Statics Under Risk

Our earlier discussions on comparative statics did not specifically take uncertainty into consideration. Whether and how would the comparative statics results carry over to choices under uncertainty?

The first result is that, if a random variable is “introduced” into a supermodular function, the expected utility function still preserves supermodularity.

Lemma 5.5.1: Expected Utility Function Preserves Supermodularity

Let $X \subset \mathbb{R}^n$ be a sublattice and $T \subset \mathbb{R}$. Suppose $u : X \times T \to \mathbb{R}$ is supermodular in $\boldsymbol{x}$. Then for any distribution function F on T, the function $U : X \to \mathbb{R}$ defined by

$$U(\boldsymbol{x}) = \int_t u(\boldsymbol{x}, t)\, \mathrm{d}F(t)$$

is supermodular in $\boldsymbol{x}$.

Proof for Lemma

Take any $\boldsymbol{x}, \boldsymbol{x}' \in X$. we have

$$\begin{aligned} U(\boldsymbol{x} \wedge \boldsymbol{x}') + U(\boldsymbol{x} \vee \boldsymbol{x}') &= \int_t [u(\boldsymbol{x} \wedge \boldsymbol{x}', t) + u(\boldsymbol{x} \vee \boldsymbol{x}', t)]\, \mathrm{d}F(t) \\ &\geq \int_t [u(\boldsymbol{x}, t) + u(\boldsymbol{x}', t)]\, \mathrm{d}F(t) \\ &= U(\boldsymbol{x}) + U(\boldsymbol{x}') \end{aligned}$$

where the inequality in the second line follows from the supermodularity of $u(\boldsymbol{x}, t)$ in $\boldsymbol{x}$. Consequently by definition of supermodularity, $U(\cdot)$ is supermodular in $\boldsymbol{x}$. ∎

The second result states that, first-order stochastic dominance preserves increasing differences.

Lemma 5.5.2: First-Order Stochastic Dominance Preserves Increasing Differences

Let $X \subset \mathbb{R}^n$ be a sublattice and $T, \Theta \subset \mathbb{R}$. Suppose $u : X \times T \to \mathbb{R}$ has increasing differences in $(\boldsymbol{x}, t)$, and $\{F_\theta\}_{\theta \in \Theta}$ is a family of distribution function on T such that $F_\theta \geq_{FOSD} F_{\theta'}$ if $\theta > \theta'$. Then the function $U : X \times \Theta \to \mathbb{R}$ defined by

$$U(\boldsymbol{x}, \theta) = \int_t u(\boldsymbol{x}, t)\, \mathrm{d}F_\theta(t)$$

has increasing differences in $(\boldsymbol{x}, \theta)$.

Proof for Lemma

Take any $\boldsymbol{x} > \boldsymbol{x}'$ and $\theta > \theta'$, we have

$$\begin{aligned} U(\boldsymbol{x}, \theta) - U(\boldsymbol{x}', \theta) &= \int_t [u(\boldsymbol{x}, t) - u(\boldsymbol{x}', t)]\, \mathrm{d}F_\theta(t) \\ &\geq \int_t [u(\boldsymbol{x}, t) - u(\boldsymbol{x}', t)]\, \mathrm{d}F_{\theta'}(t) \\ &= U(\boldsymbol{x}, \theta') - U(\boldsymbol{x}', \theta') \end{aligned}$$

where the inequality holds because, $\delta(t) := u(\boldsymbol{x}, t) - u(\boldsymbol{x}', t)$ is non-decreasing in t by increasing differences of $u(\cdot, \cdot)$ in $(\boldsymbol{x}, t)$ and $F_\theta \geq_{FOSD} F_{\theta'}$. Consequently, $U(x, \theta)$ has increasing differences in $(\boldsymbol{x}, \theta)$. ∎

Intuitively, θ can be interpreted as a signal of t, indicating the distribution of t.

Proposition 5.5.3: Comparative Statics Under Risk

Let $X \subset \mathbb{R}^n$ be a sublattice and $T, \Theta \subset \mathbb{R}$. Suppose $u : X \times T \to \mathbb{R}$ is supermodular in $\boldsymbol{x}$ and has increasing differences in $(\boldsymbol{x}, t)$. Further suppose $\{F_\theta\}_{\theta \in \Theta}$ is a family of distribution function on T such that $F_\theta \geq_{FOSD} F_{\theta'}$ if $\theta > \theta'$. Then for the function $U : X \times \Theta \to \mathbb{R}$ defined by

$$U(\boldsymbol{x}, \theta) = \int_t u(\boldsymbol{x}, t)\, \mathrm{d}F_\theta(t),$$

the set of maximizers, $\arg\max_{\boldsymbol{x} \in X} U(\boldsymbol{x}, \theta)$, is non-decreasing in θ in the strong set order.

Proof for Proposition.

- Fixing any θ, by the lemma of expected utility function preserving supermodularity, $U(\boldsymbol{x}, \theta)$ is supermodular in $\boldsymbol{x}$.
- By the lemma of first-order stochastic dominance preserving increasing differences, $U(\boldsymbol{x}, \theta)$ has increasing differences in $(\boldsymbol{x}, \theta)$.
- Since $X \times \Theta$ forms a product set, $U(\boldsymbol{x}, \theta)$ is supermodular in $(\boldsymbol{x}, \theta)$.
- By the multivariate Topkis' theorem, the set of maximizers, $\arg\max_{\boldsymbol{x} \in X} U(\boldsymbol{x}, \theta)$, is non-decreasing in θ in the strong set order.

Example.

Suppose a monopolist faces an uncertain demand and must make a production decision prior to learning the realized demand for its product. The monopolist does learn some information about demand prior to choosing its output. Formally, suppose the inverse demand function is $p(q, t) = \hat{p}(q) + t$, where $\hat{p}(q)$ is the estimated inverse demand, and t is a random noise. The ex-post profit of the firm is therefore

$$\pi(q, t) = p(q, t) \cdot q - c(q)$$

with $c(q)$ being the monopolist's cost function. The monopolist observes a signal θ that is informative about the parameter t. Specifically, suppose the distribution of t conditional on θ is $F_\theta(\cdot)$, and $F_\theta \geq_{FOSD} F_{\theta'}$ for $\theta > \theta'$. Determine how would the monopolist's optimal output $q^*(\theta)$ change with the observed signal θ.

Solution.

The monopolist solves the following maximization problem:

$$\max_{q \geq 0} \Pi(q, \theta) = \int_t \pi(q, t)\, \mathrm{d}F_\theta(t)$$

Notice that $\frac{\partial \pi(q,t)}{\partial t} = q$, increasing in q. It follows that $\pi(q, t)$ has increasing differences in (q,t). Since $F_\theta \geq_{FOSD} F_{\theta'}$ for $\theta > \theta'$, $\Pi(q,\theta)$ has increasing differences in (q,θ) and $q^*(\theta)$ is non-decreasing in θ in the strong set order.

Example.

Consider an agent with an initial wealth w, who faces a random loss $\tilde{l} \in [0,w]$. To counter the potential loss, the agent may purchase any fraction of insurance $y \in [0,1]$. Specifically, if the agent purchases y unit of insurance, then they pay a total price of yp upfront, and get paid yl if they incur a loss of l. Suppose the agent is an expected utility maximizer and has a utility function $u(x)$ for x amount of money for sure. Further suppose $u(\cdot)$ is twice continuously differentiable with $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

1. First suppose $p \leq \mathbb{E}\left[\tilde{l}\right]$. Show the optimal amount of insurance $y^* = 1$.
2. Next suppose $p > \mathbb{E}\left[\tilde{l}\right]$. Show the optimal amount of insurance $y^* < 1$.
3. Let $A(x,u)$ be the coefficient of absolute risk aversion. Write down its expression and show that if $A(x,u) = c_0$ (a constant), then the optimal amount of insurance y^* is independent of w.
4. Now suppose $A(x,u)$ strictly decreases with x. Show the optimal amount of insurance y^* is non-increasing in the agent's initial wealth w.

Solution.

1. Since the agent is an expected utility maximizer, their choice problem is given by

$$\max_{y\in[0,1]} U(y) = \mathbb{E}\left[u\left(w - yp - (1-y)l\right)\right]$$

Simple calculation shows

$$U'(y) = \mathbb{E}\left[(l-p)\cdot u'\left(w - l + (l-p)y\right)\right]$$
$$U''(y) = \mathbb{E}\left[(l-p)^2\cdot u''\left(w - l + (l-p)y\right)\right] \leq 0$$

It follows that $U'(y) \geq U'(1) = u'(w-p)\left(\mathbb{E}\left[\tilde{l}\right] - p\right) \geq 0$, with strict inequality as long as $p < \mathbb{E}\left[\tilde{l}\right]$. Consequently, $y^* = 1$.

2. When $p > \mathbb{E}\left[\tilde{l}\right]$, $U'(1) = u'(w-p)\left(\mathbb{E}\left[\tilde{l}\right] - p\right) < 0$, and $U''(y) \leq 0$, we have $y^* < 1$.

3. By definition, $A(x,u) = -\dfrac{u''(x)}{u'(x)}$.

Take any $w_1 < w_2$, and let

$$v_1(x) = u(w_1 + x)$$
$$v_2(x) = u(w_2 + x)$$

Then at initial wealth w_i $(i = 1, 2)$, the agent's maximization problem is given by

$$\max_{y \in [0,1]} V_i(y) = \mathbb{E}\left[v_i\left(-l + (l-p)\, y\right)\right]$$

From the previous two questions, it suffices to focus on the case of $p > \mathbb{E}\left[\tilde{l}\right]$.

By earlier characterization, $A(x, u) = c_0$ implies $\dfrac{v_1'(x)}{v_2'(x)} = c_1$, which is also a constant.

Simple calculation shows

$$\begin{aligned} V_i'(y) &= \mathbb{E}\left[(l-p)\, v_i'\left(-l + (l-p)\, y\right)\right] \\ V_i''(y) &= \mathbb{E}\left[(l-p)^2\, v_i''\left(-l + (l-p)\, y\right)\right] \le 0 \end{aligned}$$

If $V_1'(0) \le 0$, then $V_2'(0) \le 0$, so $y_1^* = y_2^* = 0$. Otherwise, if $V_1'(y^*) = 0$, then $V_2'(y^*) = 0$; by $V_i''(\cdot) < 0$ for $i = 1, 2$, $V_i'(y)$ strictly decreases with y and achieves maximum at y, so $y_1^* = y_2^* = y^*$.

4. Suppose by contradiction that $y_2^* > y_1^* \ge 0$. By optimality of y_2^*,

$$\int_0^w v_2'\left(-l + (l-p)\, y_2^*\right)(l-p)\, \mathrm{d}F(l) = 0.$$

By our earlier characterization, $A(x, u)$ strictly decreases with x implies $\dfrac{v_1'(x)}{v_2'(x)}$ is non-increasing in x.

Consequently, we have

$$\begin{aligned} V_1'(y_2^*) &= \int_0^p v_1'\left(-l + (l-p)\, y_2^*\right)(l-p)\, \mathrm{d}F(l) \\ &\quad + \int_p^w v_1'\left(-l + (l-p)\, y_2^*\right)(l-p)\, \mathrm{d}F(l) \\ &= \int_0^p \frac{v_1'\left(-l + (l-p)\, y_2^*\right)}{v_2'\left(-l + (l-p)\, y_2^*\right)} v_2'\left(-l + (l-p)\, y_2^*\right)(l-p)\, \mathrm{d}F(l) \\ &\quad + \int_p^w \frac{v_1'\left(-l + (l-p)\, y_2^*\right)}{v_2'\left(-l + (l-p)\, y_2^*\right)} v_2'\left(-l + (l-p)\, y_2^*\right)(l-p)\, \mathrm{d}F(l) \\ &\ge \frac{v_1'(-p)}{v_2'(-p)} \int_0^w v_2'\left(-l + (l-p)\, y_2^*\right)(l-p)\, \mathrm{d}F(l) \\ &= 0 \end{aligned}$$

Given $V_i''(\cdot) < 0$ and our assumption of $y_2^* > y_1^* \ge 0$, we have $V_1'(y_1^*) > V_1'(y_2^*) \ge 0$, which is a contradiction.

Chapter 6

General Equilibrium

The unifying idea across all market analyses is *market clearing*. What distinguishes the two standard approaches is how many markets they ask to clear at once. *Partial equilibrium* fixes prices in all other markets and analyzes a single market in isolation. *General equilibrium* requires every market in the economy to clear simultaneously, taking into account the cross-market price linkages.

Partial equilibrium remains a useful shortcut despite its narrower scope. It is a defensible approximation whenever:

- Prices in all other markets are essentially unaffected by what happens in the market of interest (e.g., the market is small relative to the rest of the economy).
- There are no significant wealth effects feeding back from the market of interest into demand elsewhere.

The following example shows what can go wrong when these conditions fail.

Example.

There are n towns in total, with n being a large number. Each town has an identical price-taking firm that produces a single consumption good with the production function $y = f(l)$, where $f'(l) > 0$ and $f''(l) < 0$. The single consumption good is traded in the national market. Suppose there are L units of inelastic labor supply in total. Workers can move freely across towns to seek the highest possible wage, which are regarded as complete information. Normalize $p_c = 1$ and let w_i denote the equilibrium wage in town i. Now suppose that town 1 levies a **small** tax $t > 0$ on firm 1 for each unit of labor hired. Which group of individual(s) will bear the tax burden?

Solution.

- Equilibrium without tax
  - Whether we adopt the partial or general equilibrium approach, the competitive equilibrium without the labor tax is identical.
  - Since workers can move freely and $f'(l) > 0$ and $f''(l) < 0$, we must have

$$\begin{cases} l^* = \frac{1}{n} \cdot L \\ w_1 = w_2 = \cdots = w_n = f'(l^*) = f'\left(\frac{l}{n}\right) \end{cases}$$

- Partial equilibrium in town 1 with tax
  - Assuming the wage rate in other markets are unaffected, we must have $w_1(t) = w_0 + t$.
  - At the new equilibrium, $f'(l_1(t)) = w_1(t) = w_0 + t$.
  - Hence we can see that firm 1 bears all the tax burden.
- General equilibrium with town 1 levied with tax
  - Let $l_1(t)$ denote the equilibrium amount of labor in town 1 with a t unit tax, and $l_{-1}(t)$ denote the equilibrium amount of labor in any other town when town 1 imposes a t unit tax and $w(t)$ the equilibrium wage rate received by the worker. When all labor markets clear, we should have:

$$\begin{cases} l_1(t) + (n-1)\, l_{-1}(t) = L \\ f'(l_1(t)) = w_1(t) = w(t) + t \\ f'(l_{-1}(t)) = w(t) \end{cases}$$
$$\Longrightarrow \begin{cases} f'(L - (n-1)\, l_{-1}(t)) = w(t) + t \\ f'(l_{-1}(t)) = w(t) \end{cases}$$
$$\Longrightarrow \begin{cases} -(n-1)\, l'_{-1}(t) \cdot f''(L - (n-1)\, l_{-1}(t)) = w'(t) + 1 \\ l'_{-1}(t)\, f''(l_{-1}(t)) = w'(t) \end{cases}$$

  - Set $t \to 0$, so we have $l_{-1}(0) = \frac{1}{n} \cdot L$. From the two equations above we obtain $w'(0) = -\frac{1}{n}$.
  - Let $\Pi(w(t))$ denote the total profit of the firms when the wage rate is $w(t)$ and $\pi(w)$ the profit of a firm paying wage w. Naturally, $\Pi(w(t)) = \pi(w(t) + t) + (n-1)\, \pi(w(t))$, and

$$\frac{\partial \Pi(w(t))}{\partial t} = (w'(t) + 1) \cdot \pi'(w(t) + t) + (n-1)\, w'(t) \cdot \pi'(w(t))$$
$$\Longrightarrow \left.\frac{\partial \Pi(w(t))}{\partial t}\right|_{t \to 0} = (w'(0) + 1) \cdot \pi'(w(0)) + (n-1)\, w'(0) \cdot \pi'(w(0)) = 0$$

  - From quantitative analysis we can see that, the firms as a whole do not bear the tax burden and the workers bear all the tax burden.
  - Intuitively, it must be the workers that bear all the tax burden. Even though the labor supply for any firm is perfectly elastic, the **total** labor supply is **perfectly inelastic**. Small as the impact on the wages in other towns, it is not negligible. Indeed, the labor supply for any given firm is perfectly elastic, so we cannot ignore the general equilibrium effect.

6.1 Pure Exchange Economy

A full market analysis typically involves three activities: consumption, production, and trade. The cleanest setting to develop general-equilibrium intuition strips out production: in a *pure exchange economy*, each agent starts with an endowment of goods, and the only economic activity is mutually beneficial trade.

Example.

Consider a perfectly competitive economy with two agents ($i = A, B$) and two goods ($j = 1, 2$). Suppose the agents' preference relations and initial endowments are given by

$$u^A(x_1, x_2) = x_1^2 x_2 \text{ with } e^A = (1, 2)$$
$$u^B(x_1, x_2) = x_1 x_2^2 \text{ with } e^B = (2, 1)$$

Can we come up with a price vector (p_1, p_2) that clear both markets?

Solution.

The key idea is that, each agent has their own income, which is endogenously given by the equilibrium price vector.

Suppose there is an equilibrium price vector (p_1, p_2), then agent A's "income" is $m^A = p_1 + 2p_2$ and agent B's "income" is $m^B = 2p_1 + p_2$. From this we can pin down the optimal individual demands

$$\boldsymbol{x}^A = \left(\frac{2(p_1 + 2p_2)}{3p_1}, \frac{1(p_1 + 2p_2)}{3p_2}\right)$$
$$\boldsymbol{x}^B = \left(\frac{1(2p_1 + p_2)}{3p_1}, \frac{2(2p_1 + p_2)}{3p_2}\right)$$

In equilibrium, market demand equals market endowment for either good, respectively:

$$\frac{2(p_1 + 2p_2)}{3p_1} + \frac{1(2p_1 + p_2)}{3p_1} = 3$$
$$\frac{1(p_1 + 2p_2)}{3p_2} + \frac{2(2p_1 + p_2)}{3p_2} = 3$$

From either equation, we have $\frac{p_1^*}{p_2^*} = 1$. (Otherwise if there comes with a contradiction, the equilibrium cannot exist.)

From this example, we obtain four observations in this two-agent, two-good pure exchange economy:

1. There is an equilibrium price vector that clears both markets.
2. Only the relative price matters in equilibrium.
3. When one market clears, the other market clears simultaneously (and miraculously).
4. The equilibrium allocation is Pareto efficient.

6.1.1 Setups

Model Preliminaries

- $I = \{1, 2, \cdots, n\}$ agents and $J = \{1, 2, \cdots, m\}$ goods.
- $\succsim^i$: preference relation of agent i.
- $\mathbf{e}^i = \left\{e_1^i, e_2^i, \cdots, e_m^i\right\}$ initial endowment of agent i (property rights are well-defined and no co-ownership).
- Denote the economy $\mathcal{E} = \left\{\succsim^i, \mathbf{e}^i\right\}_{I \in \mathcal{I}}$.

Assumption 6.1.1: Market Structure

1. Perfect and complete information.
2. Perfectly competitive markets.
   - Agents are price-takers.
   - Prices are linear.
3. Goods are perfectly divisible.

Assumption 6.1.2: Agents' Preferences and Endowments

For any agent $i \in \mathcal{I}$:

1. The preference relation $\succsim^i$ is rational (complete and transitive) and continuous.
2. The preference relation $\succsim^i$ is monotonic.
3. The preference relation $\succsim^i$ is (weakly) convex.
4. $e_j^i > 0$, for all $j \in \mathcal{J}$.

Notice that by the first assumption, the utility representation of $\succsim^i$, say $u^i(\cdot)$, is guaranteed for any $i \in I$, so we can alternatively denote the economy as $\mathcal{E} = \left(u^i(\cdot), \boldsymbol{e}^i\right)_{i \in \mathcal{I}}$.

6.1.2 Walrasian Equilibrium

Definition 6.1.3: Walrasian Equilibrium

A *Walrasian Equilibrium* for a perfectly competitive pure exchange economy $\mathcal{E} = \left(u^i(\cdot), \mathbf{e}^i\right)_{i\in\mathcal{I}}$ is a price vector $\boldsymbol{p} \geq \mathbf{0}$ and an allocation $\left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}$ such that:

1. Utility maximization: Agent $i \in \mathcal{I}$ solves:

$$\max_{\boldsymbol{x}^i \geq \mathbf{0}} u^i\left(\boldsymbol{x}^i\right) \text{ s.t. } \boldsymbol{p}\cdot\boldsymbol{x}^i \leq \boldsymbol{p}\cdot\mathbf{e}^i$$

2. Market clearing for all goods:

$$\sum_{i\in\mathcal{I}} \boldsymbol{x}^i(\boldsymbol{p}) = \sum_{i\in\mathcal{I}} \mathbf{e}^i$$

Under Walrasian equilibrium, all markets clear at the same time. A Walrasian equilibrium specifies both *equilibrium price vector* and *equilibrium allocation*.

For simplicity, suppose for any $i \in \mathcal{I}$, $\boldsymbol{x}^i(\boldsymbol{p})$ is always unique. This is guaranteed if we assume each $\succsim_i$ is strictly convex and $\boldsymbol{p} \gg \mathbf{0}$.

Definition 6.1.4: Excess Demand

The (aggregate) *excess demand* for good j is given by

$$z_j(\boldsymbol{p}) = \sum_{i\in\mathcal{I}} x_j^i(\boldsymbol{p}) - \sum_{i\in\mathcal{I}} e_j^i$$

With this formulation, a Walrasian equilibrium is achieved when each agent maximize their utility subject to budget constraint prescribed by their endowment, and

$$z_j(\boldsymbol{p}) = 0, \forall j \in \mathcal{J}$$

Proposition 6.1.5: Properties of Excess Demand

Let $\mathcal{E}$ be a perfectly competitive pure exchange economy with each agent's preference relation being rational, continuous and monotonic, then the aggregate excess demand function $\boldsymbol{z}(\boldsymbol{p})$ satisfies:

1. **Homogeneity of degree** 0: $z_j(\boldsymbol{p})$ is homogeneous of degree 0 in $\boldsymbol{p}$, for any $j \in \mathcal{J}$.
2. **Walras' law**: $\boldsymbol{p}\cdot\boldsymbol{z}(\boldsymbol{p}) = 0$.

Proof for Proposition.

- Homogeneous of degree 0
  - $\boldsymbol{x}^i(\boldsymbol{p})$ solves $\max_{\boldsymbol{x}^i\geq\mathbf{0}} u^i\left(\boldsymbol{x}^i\right)$ s.t. $\boldsymbol{p}\cdot\boldsymbol{x}^i \leq \boldsymbol{p}\cdot\boldsymbol{e}^i$.
  - For any $t > 0$, the solution must be the same as: $\max_{\boldsymbol{x}^i\geq\mathbf{0}} u^i\left(\boldsymbol{x}^i\right)$ s.t. $(t\boldsymbol{p})\cdot\boldsymbol{x}^i \leq$

$(t\boldsymbol{p}) \cdot \boldsymbol{e}^i$. Hence, $\boldsymbol{x}^i(\boldsymbol{p}) = \boldsymbol{x}^i(t\boldsymbol{p})$.

– For any $t > 0$, $z_j(t\boldsymbol{p}) = \sum_{i=1}^n x_j^i(t\boldsymbol{p}) - \sum_{i=1}^n e_j^i = \sum_{i=1}^n x_j^i(\boldsymbol{p}) - \sum_{i=1}^n e_j^i = z_j(\boldsymbol{p})$.

- Walras' law

  – Budget constraint must be binding for any agent $i \in \mathcal{I}$, $\boldsymbol{p} \cdot \boldsymbol{x}^i(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{e}^i$. (This must hold because of assumption of monotonicity of preference relation, which is stronger than locally non-satiation).

  – It follows that

$$\begin{aligned} \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) &= \boldsymbol{p} \cdot \left(\sum_{i=1}^n \boldsymbol{x}^i(\boldsymbol{p}) - \sum_{i=1}^n \boldsymbol{e}^i \right) \\ &= \sum_{i=1}^n \left(\boldsymbol{p} \cdot \boldsymbol{x}^i(\boldsymbol{p}) - \boldsymbol{p} \cdot \boldsymbol{e}^i \right) \\ &= 0 \end{aligned}$$

The following proposition, the existence of Walrasian equilibrium, is characterized as the cornerstone of modern economics.

Proposition 6.1.6: Existence of Walrasian Equilibrium

Let $\mathcal{E} = \left(u^i(\cdot), \boldsymbol{e}^i\right)_{i \in \mathcal{I}}$ be a perfectly competitive pure exchange economy that satisfies all the four assumptions on agents' preferences and endowments. Then a Walrasian equilibrium exists.

Proof for Proposition.

We give proof for the case of $m = 2$ and $\succsim^i$ being strictly convex for any $i \in \mathcal{I}$. Given rationality, continuity and strict convexity of $\succsim^i$, $\boldsymbol{x}^i(p_1, p_2)$ is always unique for any $\boldsymbol{p} \gg \boldsymbol{0}$. When $m = 2$, Walras' law states that

$$p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) = 0.$$

Then a Walrasian equilibrium is given by $z_1(p_1, p_2) = 0$ or $z_2(p_1, p_2) = 0$. By monotonicity and strictly convexity of preference relation, the only possibility is $p_1, p_2 > 0$ in equilibrium, so we can normalize $p_1 = 1$ or $p_2 = 1$.

By theorem of the maximum, given rationality, continuity, monotonicity and strict convexity of preference relation, $x_i^1(p_1, 1)$ is continuous for any $i \in \mathcal{I}$. Hence, $z_1(p_1, 1)$ is continuous on $(0, +\infty)$. Given monotonicity and strict convexity, $u^i(\cdot)$ is strongly monotone. Therefore, $\lim_{p_1 \to 0+} x_1^i(p_1, 1) = +\infty$ and $\lim_{p_1 \to 0+} z_1(p_1, 1) = +\infty$. Symmetrically, we have $\lim_{p_2 \to 0+} x_2^i(1, p_2) = +\infty$ and $\lim_{p_2 \to 0+} z_2(1, p_2) = +\infty$. By continuity, $\exists 0 < \underline{p} < \bar{p}$ such that $z_1\left(\underline{p}, 1\right) > 0 > z_1(\bar{p}, 1)$. By the intermediate value theorem, $\exists p_1^* \in \left(\underline{p}, \bar{p}\right)$ such that $z_1(p_1^*, 1) = 0$.

6.2 Allocation

Recall the two cornerstone results from intermediate micro:

- *First Welfare Theorem.* Any Walrasian equilibrium allocation is Pareto efficient.
- *Second Welfare Theorem.* Any Pareto efficient allocation can be supported as a Walrasian equilibrium after a suitable redistribution of initial endowments.

We now state these rigorously and strengthen the first. Begin with the formal definition of a feasible allocation.

Definition 6.2.1: Feasible Allocation

For a pure exchange economy $\mathcal{E}$, an allocation $\left(\boldsymbol{x}^i\right)_{i=1}^n$ with $\boldsymbol{x}^i \geq \mathbf{0}$ for all i is *feasible* if for any good j,

$$\sum_{i=1}^n x_j^i \leq \sum_{i=1}^n e_j^i.$$

In other words, total consumption of every good cannot exceed the total endowment of that good.

Definition 6.2.2: Pareto Efficient Allocation

Given an economy $\mathcal{E}$, a feasible allocation $\left(\boldsymbol{x}^i\right)_{i=1}^n$ is (strongly) *Pareto efficient* if there is no other feasible allocation $\left(\boldsymbol{w}^i\right)_{i=1}^n$ such that $\boldsymbol{w}^i \succsim^i \boldsymbol{x}^i$ for all $i \in \mathcal{I}$, with $\boldsymbol{w}^k \succ^k \boldsymbol{x}^k$ for some $k \in \mathcal{I}$.

A Pareto efficient allocation is one from which no agent can be made strictly better off without making at least one other agent strictly worse off.

Definition 6.2.3: Core Allocation

A feasible allocation $\left(\boldsymbol{x}^i\right)_{i=1}^n$ is *in the core* of $\mathcal{E}$ if there is no group $\mathcal{I}_0 \subseteq \mathcal{I}$ and an alternative allocation $\left(\boldsymbol{w}^i\right)_{i=1}^n$ such that:

1. For any $j \in \mathcal{J}$, $\sum_{i \in \mathcal{I}_0} w_j^i \leq \sum_{i \in \mathcal{I}_0} e_j^i$.
2. $\boldsymbol{w}^i \succsim^i \boldsymbol{x}^i$ for all $i \in \mathcal{I}_0$, with $\boldsymbol{w}^k \succ^k \boldsymbol{x}^k$ for some $k \in \mathcal{I}_0$.

An allocation is in the core if *no* group of agents — large or small — can find a redistribution of their own endowments that makes every member of the group at least as well off and some member strictly better off. Core allocations are Pareto efficient (taking the "grand coalition" as the blocking group recovers the Pareto condition), but Pareto efficiency does not imply the core: an allocation can be Pareto efficient overall yet have a strictly sub-coalition that could improve on it among themselves.

Theorem 6.2.4: Strengthened First Theorem of Welfare Economics

Let $\mathcal{E}$ be a perfectly competitive pure exchange economy in which each agent's preference relation is rational, continuous and concave. Denote S_P as the set of Pareto efficient allocations, S_C the set of core allocations, and S_W the set of Walrasian equilibrium allocations. Then

$$S_W \subseteq S_C \subseteq S_P$$

Proof for Theorem

By definition, $S_C \subseteq S_P$, so it suffices to show $S_W \subseteq S_C$. Suppose not, then there exists a Walrasian equilibrium $\left(\boldsymbol{p}; \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}\right)$, a group $\mathcal{I}_0 \subseteq \mathcal{I}$ and an alternative allocation $\left(\boldsymbol{w}^i\right)_{i\in\mathcal{I}}$ such that:

$$\begin{cases} \boldsymbol{w}^i \succsim^i \boldsymbol{x}^i, \text{ for all } i \in \mathcal{I}_0 \\ \boldsymbol{w}^k \succ^k \boldsymbol{x}^k, \text{ for some } k \in \mathcal{K}_0 \subseteq \mathcal{I}_0 \end{cases}$$

By direct revealed preference,

$$\begin{cases} \boldsymbol{p} \cdot \boldsymbol{w}^k > \boldsymbol{p} \cdot \boldsymbol{x}^k, \text{ for all } k \in \mathcal{K}_0 \\ \boldsymbol{p} \cdot \boldsymbol{w}^i \geq \boldsymbol{p} \cdot \boldsymbol{x}^i, \text{ for all } i \in \mathcal{I}_0 \backslash \mathcal{K}_0 \end{cases}$$

Summing over $i \in \mathcal{I}_0$, we have

$$\begin{aligned} &\boldsymbol{p} \cdot \sum_{i\in\mathcal{I}_0} \boldsymbol{w}^i > \boldsymbol{p} \cdot \sum_{i\in\mathcal{I}_0} \boldsymbol{x}^i = \boldsymbol{p} \cdot \sum_{i\in\mathcal{I}_0} \boldsymbol{e}^i \\ \Longrightarrow &\sum_{i\in\mathcal{I}_0} \boldsymbol{w}^i \nleq \sum_{i\in\mathcal{I}_0} \boldsymbol{e}^i \end{aligned}$$

∎

Remark.

- The standard First Welfare Theorem gives $S_W \subseteq S_P$; the strengthened version shows the tighter $S_W \subseteq S_C$.
- $S_W \subseteq S_C$ says that any Walrasian equilibrium is not only efficient but also *coalition-proof*: no group can do better for itself by trading internally. This is a stronger fairness justification for the price mechanism than mere Pareto efficiency.
- The theorem demands very little of preferences (rationality, continuity, concavity). What it does need is the institutional scaffolding: perfect competition, complete information, no externalities, complete markets. Drop any of these and the conclusion can fail.

Theorem 6.2.5: Second Theorem of Welfare Economics

Let $\mathcal{E}$ be a perfectly competitive pure exchange economy in which each agent's preference relation is rational, continuous and concave. If $\boldsymbol{x}^i \gg \mathbf{0}$ is Pareto efficient, for all i, then there exists an initial endowment $\left(\boldsymbol{e}^i\right)_{i\in\mathcal{I}}$ and a price vector $\boldsymbol{p} \geq \mathbf{0}$ such that $\left(\boldsymbol{p}; \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium given these endowments.

The Second Welfare Theorem is, in a sense, less operational than the First. To support a given Pareto-efficient allocation, the planner needs both the political authority to redistribute initial endowments freely *and* full information about every agent's preferences — neither of which is generally available in practice.

6.3 General Equilibrium with Production

6.3.1 Setups

Recall that in producer theory, we used production sets and ownership shares to describe the firms and their production technologies.

Suppose there are K firms ($\mathcal{K} = \{1, 2, \cdots, K\}$) in the economy, each firm $k \in \mathcal{K}$ with its production set Y^k. Let $\alpha^{ki} \geq 0$ be agent i's ownership share of firm k. The production economy can then be described as

$$\mathcal{E} = \left(\left(u^i, \boldsymbol{e}^i, \left(\alpha^{ki}\right)_{k\in\mathcal{K}}\right)_{i\in\mathcal{I}}, \left(Y^k\right)_{k\in\mathcal{K}}\right).$$

Definition 6.3.1: Walrasian Equilibrium with Production

A **Walrasian Equilibrium** for a perfectly competitive production economy $\mathcal{E} = \left(\left(u^i, \boldsymbol{e}^i, \left(\alpha^{ki}\right)_{k\in\mathcal{K}}\right)_{i\in\mathcal{I}}, \left(Y^k\right)_{k\in\mathcal{K}}\right)$ is a vector $\left(\boldsymbol{p}, \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}, \left(\boldsymbol{y}^k\right)_{k\in\mathcal{K}}\right)$ such that:

1. **Profit maximization**: Firm k solves

$$\max_{y^k \in Y^k} \boldsymbol{p} \cdot \boldsymbol{y}^k$$

2. **Utility maximization**: Agent i solves

$$\max_{\boldsymbol{x}^i \geq \mathbf{0}} u^i\left(\boldsymbol{x}^i\right) \text{ s.t. } \boldsymbol{p} \cdot \boldsymbol{x}^i \leq \boldsymbol{p} \cdot \boldsymbol{e}^i + \sum_{k\in\mathcal{K}} \alpha^{ik} \boldsymbol{p} \cdot \boldsymbol{y}^k$$

3. **Market clearing for all goods**:

$$\sum_{i\in\mathcal{I}} \boldsymbol{x}^i\left(\boldsymbol{p}\right) = \sum_{i\in\mathcal{I}} \boldsymbol{e}^i + \sum_{k\in\mathcal{K}} \boldsymbol{y}^k\left(\boldsymbol{p}\right)$$

That is, a Walrasian equilibrium specifies prices, consumption bundles, and production plans such that every agent maximizes utility (taking firm profits as given dividend income), every firm maximizes profit, and all markets clear simultaneously.

The conceptual structure is identical to the pure-exchange case; production simply adds the firms' optimization problems. For the existence theorem to go through, however, we need three regularity conditions on production technologies.

Assumption 6.3.2: Production Technology

For each firm $k \in \mathcal{K}$,

1. $Y^k \neq \varnothing$ is **closed and convex**.
2. **Shutdown and free disposal**: that is, $\mathbf{0} \in Y^k$, and $\boldsymbol{y} \in Y^k$ implies $\boldsymbol{y}' \in Y^k$, for all $\boldsymbol{y}' \leq \boldsymbol{y}$.
3. **Irreversibility**: Let $Y = \bigcup_{k\in\mathcal{K}} Y^k$, then $Y \cap (-Y) = \{\mathbf{0}\}$.

Remark.

- $Y^k \neq \varnothing$ is innocuous, otherwise since firm k is allowed to produce nothing, we can just discard Y^k if $Y^k = \varnothing$.
- Irreversibility says that the production cannot be completely reversed.

6.3.2 Existence of Walrasian Equilibrium with Production

Theorem 6.3.3: Existence of Walrasian Equilibrium with Production

Let $\mathcal{E} = \left(\left(u^i, \boldsymbol{e}^i, \left(\alpha^{ki}\right)_{k\in\mathcal{K}}\right)_{i\in\mathcal{I}}, \left(Y^k\right)_{k\in\mathcal{K}}\right)$ be a perfectly competitive production economy satisfying all the assumptions on preference relation and endowments, and those on production technology. Then a Walrasian equilibrium $\left(\boldsymbol{p}, \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}, \left(\boldsymbol{y}^k\right)_{k\in\mathcal{K}}\right)$ exists.

Remark.

- This proposition of existence is the extension of the existence result on pure exchange economies.
- The first and second theorem of welfare economics also generalize to economics with production.

Recall that when a production technology exhibits constant returns to scale, the firm's profit must be either 0 or $+\infty$. For market clearing to make sense, it must be that each firm is making 0 profit.

For simplicity, consider an economy with two goods. The production technology can linearly transform b units of good 1 into (at most) c units of good 2. The production set can then be depicted as

$$Y = \left\{ (y_1, y_2) \,|y_1 \le 0, y_2 \ge 0, y_2 \le -\frac{c}{b} y_1 \right\}.$$

Alternatively, we can present the production technology with the vector $a = \left(-1, \frac{c}{b}\right)$.

If a linear activity production $a = (a_1, a_2)$ is ever used in a Walrasian equilibrium (i.e., $\lambda^* > 0$), then it must be that (because the firm is making zero profit)

$$(p_1^*, p_2^*) \cdot (a_1, a_2) = 0.$$

Example.

Consider a perfectly competitive economy $\mathcal{E}$ with two goods (1 and 2) and two agents (A and B). The agents' utility functions and initial endowments are as follows:

$$u^A(x_1, x_2) = x_1 x_2,\ \boldsymbol{e}^A = (1, 0)$$
$$u^B(x_1, x_2) = x_1 x_2^2,\ \boldsymbol{e}^B = (0, 1)$$

First suppose the economy is pure exchange with no production.

1. Derive the set of Pareto efficient allocations.
2. Derive the set of core allocations.
3. Derive a Walrasian equilibrium $\left((p_1, p_2)\,; \left(x_1^A, x_2^A, x_1^B, x_2^B\right)\right)$.

Next we introduce production: Suppose there is a perfectly competitive firm which can transform one unit of good 1 into (at most) one unit of good 2, i.e., $\mathbf{a} = (-1, 1)$.

4. Derive a Walrasian equilibrium with production $\left((p_1, p_2,)\,; \left(x_1^A, x_2^A, x_2^B, x_2^B\right), \lambda\right)$.

Solution.

1. Since both agents' preference relations are monotonic, any Pareto efficient allocation must satisfy
$$\boldsymbol{x}^A + \boldsymbol{x}^B = \boldsymbol{e}^A + \boldsymbol{e}^B$$
Clearly, $((1,1), (0,0))$ and $((0,0), (1,1))$ are Pareto efficient.
$((1,0), (0,1))$ or $((0,1), (1,0))$ are not necessarily Pareto efficient, though we cannot make any agent better off material-wise without making the other worse off material-wise. However, agents' well-being is measured in terms of magnitude of utilities.
Apart from corner solutions, by the idea of no gain from trade in equilibrium, we must have
$$|MRS^A| = |MRS^B|$$
Therefore, the set of efficient allocations is given by
$$S_P = \left\{ \left(x_1^A, x_2^A, x_1^B, x_2^B\right) \ge \mathbf{0} : x_1^A x_2^B = 2x_2^A x_1^B, x_1^A + x_1^B = x_2^A + x_2^B = 1 \right\}.$$

It has been checked that $((1,1),(0,0))$ and $((0,0),(1,1))$ are included in the set S_P.

2. When there are only two agents, the only subgroups other than the grand coalition are $\{1\}$ and $\{2\}$. In other words, on the basis of Pareto efficiency, the additional requirement for an allocation to be in the core is that no agent is made worse in any allocation compared to their own endowment, which is also termed *individual rationality*. Since the initial endowment is the worst possible allocation for each agent, the set of core allocations $S_C = S_P$.

3. Suppose there is an equilibrium price vector (p_1, p_2). Then agent A's "income" $m^A = p_1$ and agent B's "income" $m^B = p_2$. The optimal individual demands are

$$\left(x_1^A, x_2^A\right) = \left(\frac{1}{2}, \frac{p_1}{2p_2}\right)$$
$$\left(x_1^B, x_2^B\right) = \left(\frac{p_2}{3p_1}, \frac{2}{3}\right)$$

In equilibrium, market demand equals total endowments for each good.

$$\begin{cases} \frac{1}{2} + \frac{p_2}{3p_1} = 1 \\ \frac{p_1}{2p_2} + \frac{2}{3} = 1 \end{cases} \Longrightarrow \frac{p_1^*}{p_2^*} = \frac{2}{3}$$

Hence, a Walrasian equilibrium is $\left(\boldsymbol{p} = (2,3), \boldsymbol{x}^A = \left(\frac{1}{2}, \frac{1}{3}\right), \boldsymbol{x}^B = \left(\frac{1}{2}, \frac{2}{3}\right)\right)$.

*Here "**a** Walrasian equilibrium" is emphasized because only relative prices matter in Walrasian equilibrium; theoretically there could be infinitely-many equilibria in terms of absolute prices.*

4. Suppose the production technology is used in equilibrium, then we must have $(p_1, p_2) \cdot \mathbf{a} = \mathbf{0}$, that is, $p_1 = p_2 \iff \frac{p_1}{p_2} = 1$.

In equilibrium, market demand equals market supply for each good.

$$\begin{cases} \frac{1}{2} + \frac{1}{3} = 1 - \lambda \\ \frac{1}{2} + \frac{2}{3} = 1 + \lambda \end{cases} \Longrightarrow \lambda^* = \frac{1}{6}$$

A Walrasian equilibrium is given by $\left(\boldsymbol{p}^* = (1,1), \boldsymbol{x}^A = \left(\frac{1}{2}, \frac{1}{2}\right), \boldsymbol{x}^B = \left(\frac{1}{3}, \frac{2}{3}\right), \lambda^* = \frac{1}{6}\right)$.