# Chapter 1

# Choice Theory

The utility-maximization approach to choice has several features that help explain its long and continuing dominance in economic analysis:

- **Normative usefulness.** It ties individual choices to a welfare criterion that can be compared with the government's, which underwrites both policy analysis and the modern democratic emphasis on respecting individual preferences.
- **Positive predictions.** It yields sharp comparative-statics predictions: how choices respond when prices, incomes, or other parameters change.
- **Wide scope.** The same machinery applies across consumption, labor supply, finance, and beyond.
- **Compactness.** Empirical predictions follow from a sparse model: a description of the chooser's objectives and constraints, nothing more.

## 1.1 Preference-Based Approach

Rational choice theory starts from the idea that individuals have preferences and choose so as to maximize utility, where the utility is itself a representation of those preferences. Our primary task is to formalize what that statement means and to extract from it precise predictions about the pattern of decision-making we should observe.

### 1.1.1 Preference Relation

**Definition 1.1.1: (Weak) Preference Relation**

Let $X$ be a set of possible choices. Consider a *weak preference relation* $\succsim$ over the set $X$, as a binary relationship:

$$x \succeq y \iff \text{``$x$ is at least as good as $y$''}$$

**Remark.**

> The weak preference relation $\succeq$ implies the associated strict preference relation $\succ$ and indifference relation $\sim$:
>
> - $x$ is *strictly preferred* to $y$, or $x \succ y$, if $x \succeq y$ but not $y \succeq x$.
> - $x$ is *indifferent* to $y$, or $x \sim y$, if $x \succeq y$ and $y \succeq x$.

With the weak preference relation in hand, we now impose two assumptions that together formalize what we mean by a *rational* preference.

The first is *completeness*: an agent is never undecided when faced with two choices.

### Definition 1.1.2: Completeness

A preference relation $\succeq$ on $X$ is *complete*, if for all $x, y \in X$, either $x \succeq y$, or $y \succeq x$, or both.

The second is *transitivity*: an agent's weak preference cannot cycle, except among choices to which the agent is indifferent.

### Definition 1.1.3: Transitivity

A preference relation $\succeq$ on $X$ is *transitive*, if whenever $x \succeq y$ and $y \succeq z$, $x \succeq z$.

From completeness and transitivity, we have the following corollaries:

### Corollary 1.1.4

1. While the definition of transitivity involves only 3-choice cycles, it also extends to all $n$-object cycles, i.e., that it implies that for any $n$ choices $x_1, x_2, \cdots, x_n \in X$ such that $x_1 \succeq x_2$, $x_2 \succeq x_3$, ..., $x_{n-1} \succeq x_n$, we must have $x_1 \succeq x_n$. (Hint: use induction on $n$.)
2. Transitivity of a weak preference relation $\succeq$ implies transitivity of the associated strict preference relation $\succ$ and the indifference relation $\sim$.

**Remark.**

> 1. We take the *weak* preference relation $\succsim$ as primitive (rather than the strict relation $\succ$) because completeness is the natural axiom for $\succsim$: with strict preference, we would need a third case for indifference, which is messier.
> 2. Transitivity is inconsistent with certain "framing effects" that show up in experimental data — e.g., when the order in which options are presented systematically changes the chosen alternative.

Completeness and transitivity together let us formalize rationality.

**Definition 1.1.5: Rationality**

A preference relation $\succsim$ on $X$ is *rational* if it is both *complete* and *transitive*.

### 1.1.2 Choice Rule

Given preferences, the agent's *choice rule* on an "opportunity set" $B \subseteq X$ induced by the preference relation $\succeq$ is

$$C_{\succeq}(B) = \{x \in B \mid x \succeq y,\ \forall y \in B\},$$

i.e., the set of items in $B$ the agent likes at least as much as every other alternative in $B$ — the most-preferred options.

**Remark.**

1. $C_{\succeq}(B)$ may contain more than one element — ties between equally-preferred alternatives are perfectly admissible.
2. $C_{\succeq}(B)$ can also be *empty*. The simplest sufficient condition for non-emptiness is finiteness of $B$: if $B$ is finite and non-empty, then $C_{\succeq}(B)$ is non-empty.
    - For an example where $C_{\succsim}(B)$ is empty, take the preference relation $x \succeq y \iff x \geq y$, with $X = [0, +\infty)$ and $B = (0, 1)$. The supremum 1 is not in $B$, so no element of $B$ is most preferred.
    - For infinite choice sets, we will later add technical assumptions (compactness of the choice set, continuity of the preference relation) that guarantee a choice exists.

**Proposition 1.1.6**

Suppose $\succeq$ is complete and transitive. Then, for every finite non-empty set $B$,

$$C_{\succeq}(B) \neq \varnothing.$$

***Proof for Proposition.***

Proceed by mathematical induction on the number of elements of $B$.

- $|B| = 1$, say $B = \{x\}$.
    - By completeness, $x \succeq x$, so $x \in C_{\succeq}(B)$. $C_{\succeq}(B) \neq \varnothing, \forall |B| = 1$.
- Fix $n \geq 1$ and suppose that for all sets $B$ with exactly $n$ elements, $C_{\succeq}(B) \neq \varnothing$. Next we move on to examine the case of $|B| = n + 1$.
    - Take any $B_n$ with $|B_n| = n$. By the inductive hypothesis $C_{\succeq}(B_n) \neq \varnothing$, so pick some $x^* \in C_{\succeq}(B_n)$. Let $B = B_n \cup \{x_{n+1}\}$.
    - By completeness, we have only two (not mutually-exclusive) possibilities:
        - If $x^* \succeq x_{n+1}$, then by definition $x^* \in C_{\succeq}(B)$, so $C_{\succeq}(B) \neq \varnothing$.

  - * If $x_{n+1} \succeq x^*$. Since $x^* \in C_{\succeq}(B_n)$, by definition, $x^* \succeq y, \forall y \in B_n$. By transitivity, this implies $x_{n+1} \succeq y, \forall y \in B$. Therefore, $x_{n+1} \in C_{\succeq}(B)$, so $C_{\succeq}(B) \neq \varnothing$.

- Hence, for every set $B$ with exact $n+1$ elements, $C_{\succeq}(B) \neq \varnothing$. By the principle of mathematical induction, it follows that for every finite set $B$ that is non-empty, $C_{\succeq}(B) \neq \varnothing$.

## 1.2 Choice-Based Approach

Much empirical work runs in the opposite direction. Rather than starting from preferences and predicting choices, it observes choices and tries to *rationalize* them: to determine whether the observed choices are compatible with preference maximization and, if so, what they imply about the underlying preferences. Under this approach, the choice rule is the primitive object of the theory, and preferences (if they exist) are derived from it.

### Definition 1.2.1: Choice Rule

Let $\mathcal{B}$ be the set of all nonempty subsets of $X$ ($\mathcal{B} = 2^X \backslash \varnothing = \{B \neq \varnothing : B \subset X\}$). A *choice rule* is a function $C : \mathcal{B} \to \mathcal{B}$ with the property that for all $B \in \mathcal{B}$, $C(B) \subseteq B$.

**Remark.**

- $\mathcal{B}$ is the set of all nonempty **subsets** of $X$, which means all possible set of available choice(s) the agent is facing. And the choice rule $C$ is a mapping from $\mathcal{B}$ to $\mathcal{B}$, which means that the agent is choosing from a set of available choice(s) to pick his most preferred choice(s), also a subset of $X$.

- Here we assume that we can see the agent choose from *all* possible subsets of $X$, and that the agent reports *all* of his optimal choices from a given opportunity set.

The bare definition imposes no structure on either the choice rule or any preference relation behind it. Two questions arise:

- If the rule does come from maximizing some underlying preferences, what can we infer about those preferences from the rule alone?

- Is the rule consistent with the maximization of *some* complete and transitive preference relation — i.e., is it *rationalizable*?

Consider the first question. Suppose the choice rule $C$ is consistent with maximization of some preference relation $\succsim$, i.e., $C(\cdot) = C_{\succsim}(\cdot)$. Then, observing for some $A \subseteq X$ that $y \in A$ and $x \in C(A)$ — $x$ is chosen when $y$ is available — lets us infer $x \succsim y$. This in turn implies that for any $B \subseteq X$ with $x \in B$ and $y \in C(B)$, we must also have $x \in C(B)$: we know $x \succsim y$ and $y \succsim z$ for all $z \in B$, so by transitivity $x \succsim z$ for all $z \in B$. By a symmetric argument we also get $y \in C(A)$. So any rationalizable choice rule must satisfy the following property as a *necessary* condition:

### Definition 1.2.2: HARP

A choice function $C : \mathcal{B} \to \mathcal{B}$ satisfies *Houthaker's Axiom of Revealed Preference (HARP)* if, whenever $x, y \in A \cap B$, and $x \in C(A)$ and $y \in C(B)$, we have $x \in C(B)$ and $y \in C(A)$.

In words: if $x$ and $y$ are both available in two different choice problems, and $x$ is chosen from one while $y$ is chosen from the other, then $x$ and $y$ must *both* be chosen in *both* problems. HARP rules out the obvious form of inconsistency in observed choices — picking $x$ over $y$ here and $y$ over $x$ there.

We have just shown HARP is necessary for rationalizability. The striking result is that it is also *sufficient*:

### Proposition 1.2.3

Suppose $C : \mathcal{B} \to \mathcal{B}$ is nonempty-valued. Then there exists a rational (complete and transitive) preference relation $\succsim$ on $X$ such that $C(\cdot) = C_{\succsim}(\cdot)$ if and only if $C$ satisfies HARP.

***Proof for Proposition.***

- "Only if" part: shown above.
- "If" part. Suppose $C$ satisfies HARP. We construct a rational preference relation $\succsim_C$ that rationalizes $C$.
  - *Define the revealed preference relation.* Set $x \succsim_C y$ iff there exists some $A \subseteq X$ with $y \in A$ and $x \in C(A)$. That is, $x$ is revealed preferred to $y$ whenever the agent picks $x$ from some menu that also contained $y$.
  - *$\succsim_C$ is complete.* For any $x, y \in X$, the set $C(\{x, y\})$ is non-empty, so it contains $x$, $y$, or both. In each case the corresponding ranking $x \succsim_C y$ or $y \succsim_C x$ follows by construction.
  - *$\succsim_C$ is transitive.* Suppose $x \succsim_C y$ and $y \succsim_C z$. The set $C(\{x, y, z\})$ is non-empty, so it contains at least one of $x, y, z$ (the cases are not mutually exclusive, but we just need one):
    - $x \in C(\{x, y, z\})$. Since $z \in \{x, y, z\}$, the definition of $\succsim_C$ gives $x \succsim_C z$ directly.
    - $y \in C(\{x, y, z\})$. From $x \succsim_C y$ there exists $A_0$ with $y \in A_0$ and $x \in C(A_0)$. Apply HARP with this $A_0$ and $B = \{x, y, z\}$ (both contain $x$ and $y$) to get $x \in C(\{x, y, z\})$. This reduces to case 1, so $x \succsim_C z$.
    - $z \in C(\{x, y, z\})$. From $y \succsim_C z$ there exists $A_0$ with $z \in A_0$ and $y \in C(A_0)$. HARP gives $y \in C(\{x, y, z\})$, reducing to case 2, and hence $x \succsim_C z$.
  - $C(\cdot) = C_{\succsim_C}(\cdot)$. It suffices to show that for all $A \in \mathcal{B}$ and $x \in A$, $x \in C(A) \iff x \succsim_C y$ for all $y \in A$.
    - "$\Rightarrow$" holds directly by the definition of $\succsim_C$.
    - "$\Leftarrow$" Take any $x \in C_{\succsim_C}(A)$, so $x \succsim_C y$ for all $y \in A$. Since $C(A)$ is non-empty, there is some $y_0 \in C(A)$. The definition of $\succsim_C$ gives an $A_0$ with $x, y_0 \in A_0$ and

$x \in C(A_0)$. Then $x, y_0 \in A_0 \cap A$, with $x \in C(A_0)$ and $y_0 \in C(A)$, so HARP yields $x \in C(A)$.

■

**Remark.**

1. HARP does the entire work in the "if" direction: it is precisely the axiom that endows the revealed preference relation $\succsim_C$ with the transitivity needed to call it rational.
2. The rationalizability result requires observing the *whole* choice function $C$ — that is, (i) for any given choice set, all of the agent's optimal choices are observed, not just some of them, and (ii) the agent's optimal choices are observed across *every* choice set. Real data is rarely so generous:
    - In consumer-choice problems, the menus $A$ we see are typically budget sets indexed by prices and income — a strict subcollection of $\mathcal{B}$, not all of $\mathcal{B}$.
    - The first form of incompleteness (only seeing some of the optimal choices from a given menu) is essentially intractable. For the second (only seeing some menus), other revealed-preference axioms have been developed. The *Weak Axiom of Revealed Preference (WARP)* is the natural counterpart of HARP for choice rules restricted to budget sets and required to be single-valued (so the HARP conclusion sharpens to $x = y$). WARP is necessary for rationalizability but not sufficient on budget data alone; the *Generalized Axiom of Revealed Preference (GARP)* closes the gap and is both necessary and sufficient.