Chapter 2

Consumer Theory

Choice theory says that a rational decision-maker picks the most preferred option from her choice set. Consumer theory specializes this framework to the most extensively studied application — a consumer choosing bundles of goods subject to a budget constraint — and uses the *preference-based approach* throughout. The agenda has three parts:

1. **Formalize the choice set and preferences.**
   - Set of alternatives: the *consumption set*.
   - Choice set (within the consumption set): the *budget set*.
   - Preferences: a *utility representation* of the preference relation.
2. **Derive optimal choices** from the budget set and the preference relation.
3. **Analyze the properties** of those optimal choices — i.e., of demand.

2.1 Setups

We start with four assumptions that we will maintain throughout consumer theory:

Assumption 2.1.1

1. *Perfect* information.
2. Consumers are *price takers*.
3. Prices are *linear*.
4. Goods are *divisible*.

Remark.

- *Price-taking* means the consumer treats prices $\boldsymbol{p}$ as known, fixed, and exogenous — no searching for deals, no bargaining for discounts.
- *Linear prices* means the total cost of a bundle is just $\boldsymbol{p} \cdot \boldsymbol{x}$; no quantity discounts or two-part tariffs.

- *Divisibility* is captured by working in $\mathbb{R}^n_+$. This is mostly a mathematical convenience: a discrete-good problem can still be analyzed by restricting attention to the integer points in $\mathbb{R}^n_+$, so the divisibility assumption does not rule out applications to indivisible goods.

For simplicity, we take the consumption set to be the entire non-negative orthant.

Definition 2.1.2: Consumption Set

With n goods, the *consumption set* is given by:

$$X = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \cdots, n\}.$$

The budget set is a subset of the consumption set; it further restricts the bundles the consumer can actually afford given income and prices.

Definition 2.1.3: Budget Set

With n goods and income of m, given a price vector $\boldsymbol{p} = (p_1, \cdots, p_n)' \geq 0, \boldsymbol{p} \neq \mathbf{0}$, the *budget set* is given by:

$$B(\boldsymbol{p}, m) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{x} \leq m, \boldsymbol{x} \geq \mathbf{0}\}.$$

By reinterpreting the consumption goods and the budget, the derivation of the budget set can be extended to encompass other economic problems. For example, consumption-leisure choice, and inter-temporal choice.

2.2 Utility Representation

Preference relations are intuitive but awkward to work with directly. A utility representation lets us recast preference maximization as ordinary optimization over a real-valued function — a much more tractable object.

Definition 2.2.1: Utility Representation

A preference relation $\succsim$ on X is represented by a *utility function* $u : X \to \mathbb{R}$ if

$$x \succsim y \iff u(x) \geq u(y).$$

This definition makes utility an *ordinal* object: the actual numerical values carry no economic content. Only the implied ranking matters — u encodes the consumer's relative preferences over bundles, nothing more.

If u represents $\succsim$, then the choice rule defined upon budget set is:

$$C(B(\boldsymbol{p}, m); \succsim) = \left\{\boldsymbol{x} \mid \boldsymbol{x} \text{ solves } \max_{\boldsymbol{x} \in B(\boldsymbol{p}, m)} u(\boldsymbol{x})\right\}.$$

An equivalent notation of such choice rule is $C_{\succsim}(B)$.

The natural question is whether *every* rational preference relation admits a utility representation. If the consumption set X is *finite*, the answer is yes — and the construction is direct.

Proposition 2.2.2

If X is finite, then any rational (complete and transitive) preference relation $\succsim$ on X can be represented by a utility function $u : X \to \{1, \cdots, n\}$, where $n = |X|$.

Proof for Proposition.

Proceed by induction on $|X|$.

- *Base case:* $|X| = 1$, say $X = \{x\}$. Set $u(x) = 1$. The condition holds vacuously.
- *Inductive step:* suppose any rational preference on a set of size n admits the desired representation. Take any X with $|X| = n + 1$.
  - Since $C_{\succsim}(X)$ is non-empty (Ch. 1 Proposition on finite choice sets), the complement $Y = X \setminus C_{\succsim}(X)$ has at most n elements. By the inductive hypothesis, the restriction of $\succsim$ to Y admits a utility representation $u : Y \to \{1, 2, \ldots, n\}$.
  - We extend the domain of u to X by setting $u(x) = n + 1$ for each $x \in C_{\succsim}(X)$. By construction, we have $u(x) \in \{1, \cdots, n, n + 1\}$ for all $x \in X$.
  - Now we show that the constructed u represents $\succsim$, i.e., for any $x, y \in X$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Suppose $x \succsim y$. There are three possibilities:
    - $x \in C_{\succsim}(X), y \in Y \iff u(x) = n + 1 \geq u(y)$.
    - $x, y \in C_{\succsim}(X) \iff u(x) = u(y) = n + 1$, also, $u(x) \geq u(y)$.
    - $x, y \in Y$. Then, since by construction u represents $\succsim$ on Y, $x \succsim y$ if and only if $u(x) \geq u(y)$.

■

When X is infinite the situation is more subtle. In general, a rational preference relation on $\mathbb{R}^n_+$ need not admit a utility representation $u : \mathbb{R}^n_+ \to \mathbb{R}$. The standard counter-example is lexicographic preferences.

Example.

Consider the lexicographic preferences on the square $X = \mathbb{R}^2$, where $(x_1, x_2) \succ (y_1, y_2)$ if either (1) $x_1 > y_1$ or (2) $x_1 = y_1$ and $x_2 > y_2$. These preferences cannot be represented by a utility function, and this is also an example for which "indifference curves" do not exist, because the agent is never indifferent between any two choices.

To see this, suppose by contradiction there exists a utility representation $u(x, y)$. For any $x \in \mathbb{R}_+$, consider the interval $I(x) = \big(\inf_y u(x, y), \sup_y u(x, y)\big)$. $I(x)$ is not degenerate, and $I(x_1), I(x_2)$ do not overlap for $x_1 \neq x_2$. Since rational numbers are dense, we construct a function r to pick a rational number inside $I(x)$, i.e., $r(x) \in I(x)$. Since $I(x_1), I(x_2)$ do not overlap, $r(x_1) \neq r(x_2)$ for $x_1 \neq x_2$. Thus, $r : \mathbb{R}_+ \to \mathbb{Q}_+$ is an injective function. Since $\mathbb{R}_+$ is uncountable and $\mathbb{Q}_+$ is countable, this mapping is impossible.

However, note that if we replace $X = \mathbb{R}^2_+$ with $X' = \mathbb{Q}^2_+$, the lexicographic preference

> relation would admit a utility representation. More generally, this idea is formalized into the proposition in countable set case.

For *countable* X (possibly infinite), every rational preference relation admits a utility representation — and the construction below gives one explicitly.

Proposition 2.2.3

If $X \neq \varnothing$ is countable, then any rational (complete and transitive) preference relation $\succsim$ on X can be represented by a utility function $u : X \to (0,1)$.

Proof for Proposition.

- First, construct a mapping of utility function $u : X \to (0,1)$.
  - Let $(x_n)_{n=1}^{\infty}$ be a countable enumeration of X.
  - Let $u(x_1) = \frac{1}{2}$. Consider x_{n+1}, $n \geq 1$.
    - If $x_{n+1} \sim x_i$ for some $1 \leq i \leq n$, then define $u(x_{n+1}) = u(x_i)$.
    - Otherwise, define

$$M_n = \max\{\max\{u(x_i) : x_{n+1} \succ x_i, 1 \leq i \leq n\}, 0\}$$
$$m_n = \min\{\min\{u(x_i) : x_i \succ x_{n+1}, 1 \leq i \leq n\}, 1\}$$

      By construction $M_n > m_n$. Define $u(x_{n+1}) = \frac{M_n + m_n}{2}$.
- Second, prove that $u(\cdot)$ is a utility representation of preference relation $\succsim$.
  - Take any $y, z \in X$. Since X is countable, $\exists i, j$ such that $y = x_i$ and $z = x_j$. Without loss of generality, suppose $i \leq j$.
  - By construction, $u(x_i) = u(x_j)$ if and only if $x_i \sim x_j$. For $x_j \nsim x_i$, $u(x_j) > u(x_i)$ if and only if $x_j \succ x_i$.

∎

The pathology of the lexicographic preference relation is a sudden preference reversal: $(3,3) \succ (3,2)$, yet $(3,2) \succ (x,3)$ for any x strictly less than 3 — the ranking jumps discontinuously as x crosses 3. To rule this out we impose a *continuity* restriction on preferences. Continuity is also desirable from a revealed-preference standpoint: any finite set of observed choices that is consistent with HARP can be rationalized by a continuous preference relation, so continuity costs us nothing empirically.

Definition 2.2.4: Continuity

A preference relation $\succsim$ on $X \subseteq \mathbb{R}^n$ is *continuous* if for any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \to x, y^n \to y$, and $x^n \succsim y^n$ for all n, we have $x \succsim y$.

Continuity gives us more than mere existence of a utility representation — it gives us a *continuous* one.

Proposition 2.2.5

Any complete, transitive and continuous preference relation $\succsim$ on X on $X \subseteq \mathbb{R}^n$ can be represented by a continuous utility function $u : X \to \mathbb{R}$.

Proof for Proposition.

In order to have a simple, constructive proof, we prove the proposition only for the case of a monotone preference relation $\succsim$ on $X = \mathbb{R}^n_+$.

Let $e = (1, \cdots, 1)$ and consider bundles of the form $\alpha e = (\alpha, \cdots, \alpha)$ where $\alpha \geq 0$. For each $x \in \mathbb{R}^n_+$, we construct a utility number as follows: $u(x) = \max A(x)$, where $A(x) = \{\alpha \in R_+ : \alpha e \preceq x\}$. To see that the set $A(x)$ has a maximal point, note that the set is

- Nonempty, since $0 \in A(x)$ by monotonicity of $\succsim$;
- Closed, by the continuity of $\succsim$;
- Bounded, since by monotonicity of $\succsim$, $\alpha \leq \max\{x_1, \cdots, x_n\}$ for each $\alpha \in A(x)$.

Now we show that we must have $u(x)e \sim x$.

1. $u(x)e \preceq x$. This is satisfied by construction of $u(x) \in A(x)$.
2. $u(x)e \succsim x$. For each $n \geq 1$, we have $u(x) + \frac{1}{n} \notin A(x)$, hence $(u(x) + \frac{1}{n})e \not\preceq x$, therefore by completeness of $\succsim$ we have $(u(x) + \frac{1}{n})e \succsim x$, which by continuity of $\succsim$ implies
$$\lim_{n\to\infty} (u(x) + \frac{1}{n})e = u(x)e \succsim x.$$

Now it remains to show that the constructed utility function $u(\cdot)$ has

1. Ability to represent the preference relation $\succsim$, and
2. Continuity.

For representation part, note that by transitivity $x \succsim y$ if and only if $u(x)e \succsim u(y)e$ (since $u(x)e \sim x \succsim y \sim u(y)e$), and by monotonicity of $\succsim$, this holds if and only if $u(x) \geq u(y)$. For continuity part, this is more subtle and not covered here. ■

Remark.

1. The construction in the proof specifies the utility of any bundle x by finding the point on the 45° line on the indifference curve passing through x.
   - This specification is, of course, completely arbitrary; just for mathematical convenience.
   - To reflect this arbitrariness, utility representation of preferences is *ordinal*, i.e., only the induced preference ordering of choices is meaningful, instead of the exact utility numbers assigned to them.
   - In fact, if u represents $\succsim$, then $U(\cdot) = v(u(\cdot))$ also represents $\succsim$ so long as $v : \mathbb{R} \to \mathbb{R}$ is an increasing function.

2. The thought process and the introduction of the additional assumption are the most important for this part.

   - Recall that our motivation is to come up with a tractable framework to analyze the consumer's problem. One possibility is to attach a utility level to each consumption bundle (ordinal utility representation).
   - We ask whether any preference relation can be represented by a utility function. The answer is yes, if the consumption set X is finite (or countable), but no if $X = \mathbb{R}^n_+$. We notice that the problem with the counter-example (e.g., the lexicographic preference relation) is that there are sudden preference reversals.
   - We then impose the restriction of "continuity" of preferences to rule out those unfavorable scenarios and show that any rational and continuous preference relation on $X = \mathbb{R}^n_+$ can be represented by a continuous utility function.

3. The continuity of preference relation and continuity of its utility representation is not interdependent. If a preference relation can be represented by a continuous utility function, then such preference relation must be continuous. On the other hand, a continuous preference relation can be represented by a discontinuous utility function, if you like. (That is not convenient for mathematical issues though.)

2.3 Utility Maximizing Problem

What distinguishes consumer theory from the general choice-theoretic framework is the specific structure of the consumer's choice set — it is determined by prices and wealth. This structure is what lets us derive economically meaningful comparative statics. With a utility representation $u(\cdot)$ of $\succsim$ in hand, the consumer's optimization is the *Utility Maximization Problem* (UMP), also called the *Consumer Problem* (CP).

Definition 2.3.1: Utility Maximization Problem

$$\begin{aligned} &\max_{\boldsymbol{x} \in \mathbb{R}^n_+} u(\boldsymbol{x}) \\ \text{s.t. } &\boldsymbol{p} \cdot \boldsymbol{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

That is: given prices $\boldsymbol{p} = (p_1, \dots, p_n)' \geq \mathbf{0}$, $\boldsymbol{p} \neq \mathbf{0}$ and wealth m, the consumer picks a non-negative bundle $\boldsymbol{x} = (x_1, \dots, x_n)'$ to maximize utility subject to spending no more than m. Equivalently, the CP can be written as:

$$C(B(\boldsymbol{p}, m); \succsim) = \left\{ \boldsymbol{x} \;\middle|\; \boldsymbol{x} \text{ solves } \max_{\boldsymbol{y} \in B(\boldsymbol{p}, m)} u(\boldsymbol{y}) \right\}.$$

2.3.1 Existence of Optimal Choice(s)

The first question is whether the UMP has any solution. Under mild conditions, it does.

Proposition 2.3.2

Suppose that the consumer has a rational (complete and transitive) and continuous preference relation and makes rational decisions, then for $(\boldsymbol{p}, m) \gg \mathbf{0}$, they have (at least) one optimal choice.

Proof for Proposition.

- Since the consumer has a complete, transitive and continuous preference relation, then her preferences can be represented by a continuous utility function $u(\cdot)$.
- When $(\boldsymbol{p}, m) \gg \mathbf{0}$, the budget set $B(\boldsymbol{p}, m)$ is closed and bounded and hence compact.
- A continuous function on a compact set has at least one maximizer.

2.3.2 Further Assumptions

Existence is guaranteed under rationality, continuity, and $(\boldsymbol{p}, m) \gg \mathbf{0}$. The remaining questions are practical:

- How do we actually *find* the optimal choice(s)?
- Which additional restrictions on $\succsim$ simplify the analysis?

Locally Non-Satiation One useful simplification is to replace the budget inequality with the equality $\boldsymbol{p} \cdot \boldsymbol{x} = m$. Intuitively, if the consumer always thinks "more is better," she should never leave money on the table — at any optimum, she exhausts her budget.

Definition 2.3.3: Monotonicity

A preference relation $\succsim$ on $X = \mathbb{R}^n_+$ is *monotone* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$, we have:

- $\boldsymbol{x} \geq \boldsymbol{y} \Longrightarrow \boldsymbol{x} \succsim \boldsymbol{y}$;
- $\boldsymbol{x} \gg \boldsymbol{y} \Longrightarrow \boldsymbol{x} \succ \boldsymbol{y}$.

Monotonicity is more than we need; the weaker condition of *local non-satiation* already does the job.

Definition 2.3.4: Locally Non-Satiation

A preference relation $\succsim$ on $X = \mathbb{R}^n_+$ is *locally non-satiated* if for any $\boldsymbol{x} \in X$ and $\varepsilon > 0$, there exists $\boldsymbol{y} \in Ball(\boldsymbol{x}, \varepsilon) \cap X$ such that $\boldsymbol{y} \succ \boldsymbol{x}$.

Intuitively, local non-satiation rules out any *bliss point*: from any bundle, the consumer can do strictly better with an arbitrarily small change. Note that monotonicity implies local non-satiation (but not conversely).

Proposition 2.3.5

Suppose that the consumer has a complete, transitive, continuous and locally non-satiated preference relation and makes rational decisions, then for $(\boldsymbol{p}, m) \gg \mathbf{0}$, the budget constraint must hold with equality at any optimal choice $\boldsymbol{x}^*$, i.e., $\boldsymbol{p} \cdot \boldsymbol{x}^* = m$.

Proof for Proposition.

Intuitively, if the consumer does not exhaust her budget, then there must be a nearby affordable bundle which is strictly more preferred, by locally non-satiation.

- Consider any bundle $\boldsymbol{x} \in X = \mathbb{R}^n_+$ where $\boldsymbol{p} \cdot \boldsymbol{x} < m$. Since the preference relation $\succsim$ is locally non-satiated, for any $\varepsilon > 0$, there exists $\boldsymbol{y} \in Ball(\boldsymbol{x}, \varepsilon) \cap X$ such that $\boldsymbol{y} \succ \boldsymbol{x}$.
- We claim that $\boldsymbol{y} \in B(\boldsymbol{p}, m)$, i.e., the alternative bundle falls within the consumer's budget set for $\varepsilon > 0$ small enough, so that $\boldsymbol{x}$ cannot be an optimal choice.

  Let $\boldsymbol{y} = \boldsymbol{x}+\boldsymbol{z}$. By construction, $||\boldsymbol{z}|| < \varepsilon$. In particular, $|z_i| < \varepsilon$, for any $i = 1, 2, \cdots, n$. It follows that $\boldsymbol{p} \cdot \boldsymbol{y} = \boldsymbol{p} \cdot \boldsymbol{x} + \boldsymbol{p} \cdot \boldsymbol{z} \leq \boldsymbol{p} \cdot \boldsymbol{x} + n \cdot \varepsilon \cdot \max_i p_i$. As ε take appropriately small values, we finish our proof.

Convexity Another possible simplification is the case of *unique* optimal choice. Notice that the consumer's budget set $B(\boldsymbol{p}, m)$ is convex. Based on this, when the utility function $u(\cdot)$ is *strictly quasi-concave*, there is a *unique* global maximizer. This translates into the following condition of strictly convex preference relations.

Definition 2.3.6: Convexity

A preference relation $\succsim$ on $X = \mathbb{R}^n_+$ is *convex* if for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ such that $\boldsymbol{x} \succsim \boldsymbol{z}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$, we have $t\boldsymbol{x} + (1-t)\boldsymbol{y} \succsim \boldsymbol{z}$, for any $t \in [0, 1]$. If $t\boldsymbol{x} + (1-t)\boldsymbol{y} \succ \boldsymbol{z}$ for any $\boldsymbol{x} \neq \boldsymbol{y}$ and $t \in (0, 1)$, then the preference relation is *strictly convex.*

Remark.

1. Convexity can be equivalently defined as: For any $\boldsymbol{x}, \boldsymbol{y} \in X$ such that $\boldsymbol{y} \succsim \boldsymbol{x}$, we have $t\boldsymbol{y} + (1-t)\boldsymbol{x} \succsim \boldsymbol{x}$ for any $t \in [0, 1]$.
2. An equivalent way to describe convexity involves indifference curve and *upper contour set* of choice bundles, Upper Contour Set of $y = \{x \in X : x \succsim y\}$, graphically the area sitting upper-right above the indifference curve (included). Convexity of preferences amounts to the assumption that the upper contour set of any $y \in X$, is a *convex* set.
3. Convexity is fundamental in the standard model of competitive economics. When consumer preferences are convex, market clearing prices exist; otherwise this may not exist. Convexity is also needed to be able to recover consumer preferences from choices from various budget sets. Convexity is often described as capturing the idea that the agent like diversity. However, whether convexity makes sense often depends on the interpretation of the goods space, in particular on the level of aggregation (e.g., over time or categories).

Proposition 2.3.7

Suppose that the consumer has a complete, transitive, continuous and strictly convex preference relation and makes rational decisions, then for $(\boldsymbol{p}, m) \gg \mathbf{0}$, there is exactly one optimal choice.

Proof for Proposition.

- The rational and continuous preference relation has guaranteed the existence of at least one optimal choice.
- Suppose to the contrary that the consumer has at least two optimal choices $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, and $\boldsymbol{x}^* \neq \boldsymbol{y}^*$, then by optimality $\boldsymbol{x}^* \sim \boldsymbol{y}^*$. Construct a bundle $\mathbf{w} = \frac{1}{2}\boldsymbol{x}^* + \frac{1}{2}\boldsymbol{y}^*$. Since the consumer's preference relation is strictly convex, $\boldsymbol{w} \succ \boldsymbol{x}^* \sim \boldsymbol{y}^*$. Moreover, since the budget set $B(\boldsymbol{p}, m)$ is convex when $(\boldsymbol{p}, m) \gg \mathbf{0}$, so $\mathbf{w} \in B(\boldsymbol{p}, m)$. So neither $\boldsymbol{x}^*$ nor $\boldsymbol{y}^*$ can be an optimal choice, which is a contradiction.

These assumptions on $\succsim$ each have a direct counterpart on any representing utility function u. The translation makes it easier to work in the utility space and import the right structural property without re-deriving it.

Proposition 2.3.8

Suppose the preference relation $\succsim$ on X can be represented by $u : X \to \mathbb{R}$. Then,

1. $\succsim$ is monotone if and only if u is non-decreasing.
2. $\succsim$ is locally non-satiated if and only if u has no local maxima in X.
3. $\succsim$ is (strictly) convex if and only if u is (strictly) quasi-concave.

2.3.3 Marshallian Demand & Indirect Utility Function

Definition 2.3.9: Indirect Utility Function

Given $(\boldsymbol{p}, m) \gg \mathbf{0}$, the *indirect utility function* is defined to give the optimal utility level:

$$v(\boldsymbol{p}, m) = \sup_{\boldsymbol{x} \in B(\boldsymbol{p}, m)} u(\boldsymbol{x}).$$

Here we rigorously use "sup" to define $v(\boldsymbol{p}, m)$ instead of "max", because sometimes $u(\boldsymbol{x})$ does not behave well to have a maximum.

Definition 2.3.10: Marshallian Demand Correspondence

The *Marshallian demand correspondence* is defined as the consumer's optimal choice(s):

$$\boldsymbol{x}^M(\boldsymbol{p}, m) = \{\boldsymbol{x} \in B(\boldsymbol{p}, m) : u(\boldsymbol{x}) = v(\boldsymbol{p}, m)\}.$$

In general $\boldsymbol{x}^M(\boldsymbol{p}, m)$ is a *set* of utility-maximizing bundles — it collapses to a single point only under strict convexity (see below).

Throughout this section assume the consumer has a rational preference relation $\succsim$ and faces $(\boldsymbol{p}, m) \gg \mathbf{0}$. The Marshallian demand and indirect utility function inherit the following properties from $\succsim$:

Proposition 2.3.11: Properties of Marshallian Demand Correspondence and Indirect Utility Function

- **Existence of optimal choice(s)**: If $\succsim$ is continuous, then the Marshallian demand $\boldsymbol{x}^M(\boldsymbol{p}, m) \neq \varnothing$.
- **Structure of Marshallian demand**: If $\succsim$ is convex, then $\boldsymbol{x}^M(\boldsymbol{p}, m)$ is a convex set. If $\succsim$ is strictly convex, then $\boldsymbol{x}^M(\boldsymbol{p}, m)$ Is a singleton.
- **Homogeneity**: Both $v(\boldsymbol{p}, m)$ and $\boldsymbol{x}^M(\boldsymbol{p}, m)$ are homogeneous of degree 0 in $(\boldsymbol{p}, m)$, that is, for any $t > 0$, $v(t\boldsymbol{p}, tm) = v(\boldsymbol{p}, m)$ and $\boldsymbol{x}^M(t\boldsymbol{p}, tm) = \boldsymbol{x}^M(\boldsymbol{p}, m)$.
- **Monotonicity of $v(\boldsymbol{p}, m)$**: $v(\boldsymbol{p}, m)$ is non-increasing in $\boldsymbol{p}$ and non-decreasing in m. If $\succsim$ is locally non-satiated, then $v(\boldsymbol{p}, m)$ is strictly increasing in m.
- **Walras' Law**: If $\succsim$ is locally non-satiated, then $\boldsymbol{p} \cdot \boldsymbol{x} = m$, for any $\boldsymbol{x} \in \boldsymbol{x}^M(\boldsymbol{p}, m)$.

2.3.4 Derivation of Utility Maximization Problem

The consumer's utility maximization problem is

$$\begin{aligned} &\max_{\boldsymbol{x} \in \mathbb{R}^n_+} u(\boldsymbol{x}) \\ \text{s.t. } &\boldsymbol{p} \cdot \boldsymbol{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

For $(\boldsymbol{p}, m) \gg \mathbf{0}$, the constraint qualification is always satisfied, and we can apply the necessary KKT conditions when $u(\cdot)$ is continuously differentiable.
The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu) = u(\boldsymbol{x}) + \lambda \left(m - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \mu_i x_i,$$

where λ is the Lagrange multiplier on the budget constraint and, for each i, μ_i is the multiplier on the constraint that $x_i \geq 0$. The UMP is then transformed into:

$$\max_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \lambda, \mu) = \max_{\boldsymbol{x}} u(\boldsymbol{x}) + \lambda \left(m - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \mu_i x_i.$$

The first-order conditions are given by:

- w.r.t. x_i : $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i + \mu_i = 0$.
- Inequality constraints: $m - \sum_{i=1}^{n} p_i x_i \geq 0, x_i \geq 0, \lambda \geq 0, \mu_i \geq 0$.
- Complementary slackness: $\lambda \left(m - \sum_{i=1}^{n} p_i x_i\right) = 0, \mu_i x_i = 0$.

Remark.

- In contrast to equality constraints, the direction of each inequality constraint determines the way in which we set up the Lagrangian.

  Intuitively, we make sure each multiplier is non-negative and penalize constraint violations. For instance, when the budget constraint is violated, that is, $m - \sum_{i=1}^{n} p_i x_i < 0$, the value of the Lagrangian strictly decreases.

- Despite little economic meaning, λ represents the *shadow price* of wealth, that is, marginal utility of an additional unit of income.

  However, note that utility only has ordinal meanings. Nothing in the consumer theory developed so far suggests any basis for using the shadow price of wealth to guide redistribution policies.

- If the preference relation is well-behaved (i.e., locally non-satiated and strictly convex) and the non-negativity constraints are not binding, then $\frac{\partial u}{\partial x_i} = \lambda p_i$, and we are back to the familiar "tangency conditions", that is, for all i, j:

  $$MRS_{ij} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} = \frac{p_i}{p_j}.$$

  The "tangency conditions" say that at the consumer's maximum, relative marginal utility of any two choices equals their relative price, that is, all possible inner "gains from trade" has been realized (thus no more inner "gain from trade").

To facilitate the derivation of the consumer's problem, we should **first check whether the preference relation is locally non-satiated and strictly convex**.

- Case 1: If *locally non-satiated* and *strictly convex*, then we proceed in two steps

  1. Directly apply the "tangency conditions": $\frac{MU_i}{p_i} = \frac{MU_j}{p_j} = \lambda$, for any i, j.
  2. Check the *non-negativity constraints* and apply the complementary slackness condition(s) if necessary.

     Note that in the previous step we first assume that the non-negativity holds and that we have interior solution.

- Case 2: If neither locally non-satiated nor strictly convex, then use logic or economic intuition to tackle the problem.

Cobb-Douglas Utility Example.

Suppose that the consumer's preference relation can be represented by the following utility function:

$$u(x_1, x_2, x_3) = (x_1 + a)(x_2 + b)(x_3 + c)$$

where $a, b, c \geq 0$ are non-negative constants. Moreover, the consumer faces constant prices $p_1, p_2, p_3\mathbf{0}$ and has income $m \geq 0$.

1. First suppose that $a = b = c = 0$. Solve for the consumer's Marshallian demand correspondence $\boldsymbol{x}^M(x_1, x_2, x_3)$ and indirect utility function $v(p_1, p_2, p_3, m)$.

2. Next suppose that $a, b, c > 0$. Solve for the consumer's Marshallian demand correspondence $\boldsymbol{x}^M(p_1, p_2, p_3, m)$ and indirect utility function $v(p_1, p_2, p_3, m)$.

Claim: Cobb-Douglas Utility

Cobb-Douglas utility representation for a preference relation $\succsim$ can be written as

$$u(x_1, x_2, \cdots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Then the consumer would spend her income on each good according to its share,

$$x_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} \cdot \frac{m}{p_i} \iff p_i x_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} \cdot m$$

Solution.

1. To solve the first question, we need to rigorously follow the steps:
   - Step 1:Apply the positive monotonic transformation for computation-convenience: $v = \ln u$.
   - Step 2: Check the locally non-satiation and convexity of preference relation for simplification of optimal solution.
     - Check that the preference relation is monotonic and hence locally non-satiated.
     - Check that the preference relation is strictly convex, that is, the utility function is strictly quasi-concave.
   - Step 3: Now that both non-satiation and convexity are satisfied, apply the "tangency conditions":
   $$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \frac{MU_3}{p_3}$$
   - Step 4: Together with the budget constraint, so we have the Marshallian demand correspondence $\boldsymbol{x}^M(\boldsymbol{p}, m)$.

$$\boldsymbol{x}^M(p_1, p_2, p_3, m) = \left(\frac{m}{3p_1}, \frac{m}{3p_2}, \frac{m}{3p_3}\right) \geq \mathbf{0}.$$

2. The first three steps are quite similar to the previous part and thus omitted. In step 4, this time we have

$$\boldsymbol{x}_1^M(p_1,p_2,p_3,m) = \frac{m+p_1a+p_2b+p_3c}{3p_1} - a$$
$$\boldsymbol{x}_2^M(p_1,p_2,p_3,m) = \frac{m+p_1a+p_2b+p_3c}{3p_2} - b$$
$$\boldsymbol{x}_3^M(p_1,p_2,p_3,m) = \frac{m+p_1a+p_2b+p_3c}{3p_3} - c$$

Note that depending on the parameter values, we may or may not have $\boldsymbol{x}^M(\boldsymbol{p},m) \geq \mathbf{0}$. In other words, the non-negativity constraints may be binding and are for us to check. For simplicity, suppose that

$$2p_1a - p_2b - p_3c \geq 2p_2b - p_1a - p_3c \geq 2p_3c - p_1a - p_2b$$

The other five symmetric cases are similar. In our assumption, $p_1a \geq p_2b \geq p_3c$.

(a) $m \geq 2p_1a - p_2b - p_3c$, then the non-negativity constraints are not binding, and we have the interior solution described above.

(b) $2p_1a - p_2b - p_3c \geq m$, then at optimum, $x_1^* = 0$. The constraint $x_1^M \geq 0$ is binding, so the consumer only purchases goods 2 and 3, and we have

$$\boldsymbol{x}^M(p_1,p_2,p_3,m) = \left(0, \frac{m+p_2b+p_3c}{2p_2} - b, \frac{m+p_2b+p_3c}{2p_3} - c\right)$$

i. $m \geq p_2b - p_3c$, then $\boldsymbol{x}^M$ is as above.

ii. $p_2b - p_3c > m$, then the two constraints $x_1^M \geq 0$ and $x_2^M \geq 0$ are both binding, so the consumer only purchases good 3, and

$$\boldsymbol{x}^M(p_1,p_2,p_3,m) = \left(0, 0, \frac{m}{p_3} - c\right)$$

Inter-Temporal Choice Example.

You have a saving $s > 0$ to spend for this year and next year. Since you are now in graduate school, you will not earn any additional income over the two years. Suppose your utility is time-separable and is given by $v(c_1,c_2) = u(c_1)+\beta u(c_2)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable function with $u'(c) > 0$ and $u''(c) < 0$ for any $c \in \mathbb{R}_+$ and $0 < \beta < 1$ is the discount factor. Further suppose that the going interest rate is $0 < r < 1$, which will remain constant. Let c_1^* and c_2^* be your optimal consumption choices for this year and next year.

1. Give a (necessary and sufficient) condition for $(c_1^*, c_2^*) \gg \mathbf{0}$.

2. Suppose that $(c_1^*, c_2^*) \gg \mathbf{0}$, compare c_1^* with c_2^*.

Solution.

1. The utility maximization problem is given by:

$$\max_{c_1,c_2} v(c_1,c_2) = u(c_1) + \beta u(c_2)$$
$$\text{s.t. } c_1 + \frac{c_2}{1+r} \le s$$
$$c_1, c_2 \ge 0$$

Since $u(\cdot)$ is strictly increasing and strictly concave, $v(\cdot)$ is also strictly increasing and strictly concave (in particular, strictly quasi-concave), and we can directly apply the "tangency condition":

$$\frac{u'(c_1)}{1} = \frac{\beta u'(c_2)}{\frac{1}{1+r}}$$

By strictly monotonicity, $c_1 + \frac{c_2}{1+r} = s$. For $(c_1^*, c_2^*) \gg \mathbf{0}$, we need:

$$\frac{u'(s)}{u'(0)} < \beta(1+r) < \frac{u'(0)}{u'\left((1+r)s\right)}$$

2. At the optimum,

$$\frac{u'(c_1)}{u'(c_2)} = \beta(1+r)$$

- If $\beta(1+r) > 1$, then $\frac{u'(c_1)}{u'(c_2)} > 1$ and $c_1^* < c_2^*$.
- If $0 < \beta(1+r) < 1$, then $\frac{u'(c_1)}{u'(c_2)} < 1$ and $c_1^* > c_2^*$.
- If $\beta(1+r) = 1$, then $\frac{u'(c_1)}{u'(c_2)} = 1$ and $c_1^* = c_2^*$.

Intuitively, $\beta(1+r)$ represents how your tradeoff between next year and this year compares with that of the market when $c_1 = c_2$.

Utility Representation Example.

Let $X = \mathbb{R}_+ \times \mathbb{N}$, where (x,t) is interpreted as receiving x yuan at time t. Consider the following six properties of preference relations on X:

- Rationality (completeness and transitivity).
- Continuity.
- There is indifference between receiving 0 yuan at time 0 and receiving 0 yuan at any other time.
- It is (strictly) better to receive any positive amount of money as soon as possible.
- Money is always desirable.
- The preference between (x,t) and $(y,t+1)$ is independent of t.

Consider the following questions:

1. Use precise mathematical language to formally define the six properties.

2. Suppose a preference relation on X can be represented by the utility function $v(x,t) = u(x)\beta^t$, where $0 < \beta < 1$ and $u(\cdot)$ is continuous, strictly increasing and $u(0) = 0$. Check whether this preference relation satisfies each of the six properties.

3. Suppose a preference relation on X satisfies all of the six properties. Show that this preference relation must admit a utility representation.

4. Use precise mathematical language to formalize the idea that "one preference is more patient than another".

5. Based on your definition in part (4), prove or disprove the following statement: A preference relation represented by $v_1(x,t) = u_1(x)\beta_1^t$ is more patient than another preference relation represented by $v_2(x,t) = u_2(x)\beta_2^t$ if $0 < \beta_2 < \beta_1$ (where $u_1(\cdot)$ and $u_2(\cdot)$ are both continuous, strictly increasing and $u_1(0) = u_2(0) = 0$).

Solution.

1. Mind yourself that the time t is not continuous here, so take caution when you try to define a "limit" with regard to t.

   - Continuity: For any $t, t' \in \mathbb{N}$, and any pair of sequences $\{x(n)\}_{n=1}^{\infty}$ and $\{y(n)\}_{n=1}^{\infty}$ from $\mathbb{R}^+$ with $x(n) \to x^*$, $y(n) \to y^*$. If $(x(n), t) \succsim (y(n), t')$ for all n, we have $(x^*, t) \succsim (y^*, t')$.
   - The preference between (x,t) and $(y, t+1)$ is independent of t: For any $x, y \in \mathbb{R}^+$, and $t, t' \in \mathbb{N}$, we have $(x,t) \succsim (y, t+1) \iff (x, t') \succsim (y, t'+1)$.

2. A continuous preference relation can be represented by a discontinuous function. However, if a preference relation can be represented by a continuous function, then the preference relation must be continuous.

3. Recall the proof in our lecture of existence of utility representation for any rational and continuous preference relation.

   - Claim 1: For any pair (x,t), there is a unique number $u(x,t) \in \mathbb{R}^+$ such that $(x,t) \sim (u(x,t), 0)$.

     Proof: First, by indifference when receiving nothing, we have $(0,t) \sim (0,0)$. For any pair (x,t), we have $(x,t) \succsim (0,0)$ and $(x+1, t) \succsim (x,t)$. Then by continuity, there is y for which $(x,t) \sim (y, 0)$, and we then define $u(x,t) := y$.
   - Claim 2: The preference relation is represented by $u(x,t)$.

     By claim 1 and property that money is more desirable:

$$u(x,t) \geq u(y,t') \iff (u(x,t), 0) \succsim (u(y,t'), 0) \iff (x,t) \succsim (y,t')$$

4. The definition should be clearly based on preference relations and try to be somewhat math-irrelevant.

   $\succsim_1$ is more patient than $\succsim_2$ if for any (x,t) and any (y,t') with $t' > t$, $y > x$:

$$(y,t') \succsim_2 (x,t) \Longrightarrow (y,t') \succsim_1 (x,t)$$

The definition means that if I prefer to wait from t to t' under $\succsim_1$, then I will also prefer to wait from t to t' when I'm more patient (say under $\succsim_2$).

5. Intuitively, if the two preference relations value money differently at the baseline level (i.e., simply in terms of money), they would generate different preference over combinations of money and receiving time.

Example.

Let $\succsim$ be a rational (complete and transitive) preference relation on $X = \mathbb{R}^2_+$. Consider the following three properties:

- Additivity: If $(x_1, x_2) \succsim (y_1, y_2)$, then for any t, s such that $(x_1+t, x_2+s), (y_1+t, y_2+s) \in \mathbb{R}^2_+$, $(x_1+t, x_2+s) \succsim (y_1+t, y_2+s)$.
- Strong monotonicity: If $x_1 \geq y_1$ and $x_2 \geq y_2$, then $(x_1, x_2) \succsim (y_1, y_2)$. If in addition, $x_1 > y_1$ or $x_2 > y_2$, then $(x_1, x_2) \succ (y_1, y_2)$.
- Standard continuity: For any two sequences $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ and $\{\boldsymbol{y}_n\}_{n=1}^{\infty}$, if $\boldsymbol{x}_n \succsim \boldsymbol{y}_n$ for any n, and $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^*$ and $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}^*$, then $\boldsymbol{x}^* \succsim \boldsymbol{y}^*$.

Consider the following questions:

1. Show that if $\succsim$ has a linear utility representation, i.e., $u(x_1, x_2) = ax_1 + bx_2$, for some $a, b > 0$, then this preference relation satisfies the above three properties.
2. Show that these three properties are necessary for the preference relation $\succsim$ to have a linear utility representation, i.e., show that for any pair of the three properties, there is a preference relation that does not satisfy the third property.
3. Show that if $\succsim$ satisfies the three properties, then this preference relation admits a linear utility representation, i.e., there exists $a, b > 0$ such that $u(x_1, x_2) = ax_1 + bx_2$, for any $(x_1, x_2) \in \mathbb{R}^2_+$. (Hint: Think about the indifference curves/sets of this preference relation.)

Solution.

1. Easy to verify. Notice that if a preference relation can be represented by a continuous function, then the preference relation must be continuous.
2. This question means the three properties are "parallel" from a preference relation to have a linear utility representation.
   - (i)(ii)$\overset{\times}{\Longrightarrow}$(iii): The lexicographic preference: $(x_1, x_2) \succsim (y_1, y_2)$ if either $x_1 > x_2$ or $x_1 = x_2$ and $y_1 > y_2$.
   - (i)(iIi)$\overset{\times}{\Longrightarrow}$(ii): $u(x_1, x_2) = x_1 - x_2$; $u(x_1, x_2) = x_1 - \frac{1}{x_2}$.
   - (iI)(iIi)$\overset{\times}{\Longrightarrow}$(i): $u(x_1, x_2) = x_1^2 + x_2^2$; $u(x_1, x_2) = x_1$.
3. Starting from possible intuitions from linear utility representation, we need to establish the following two properties of the indifference curve:

- Property 1: The indifference curves are linear.
- Property 2: The indifference curves are parallel, downward sloping and not thick.

In order to establish the two properties, we first prove the following two lemmas:

- Lemma 1: For $\boldsymbol{x} \neq \boldsymbol{y}$, if $\boldsymbol{x} \sim \boldsymbol{y}$, then for $\boldsymbol{z} = \frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}$ and $\boldsymbol{z}' = 2\boldsymbol{y} - \boldsymbol{x}$, we have $\boldsymbol{x} \sim \boldsymbol{y} \sim \boldsymbol{z} \sim \boldsymbol{z}'$.
- Lemma 2: For $\boldsymbol{x} \neq \boldsymbol{y}$, if $\boldsymbol{x} \sim \boldsymbol{y}$, then for any $\boldsymbol{w} = t\boldsymbol{x} + (1-t)\boldsymbol{y}, 0 \leq t \leq 1$, we have $\boldsymbol{x} \sim \boldsymbol{y} \sim \boldsymbol{w}$.

Here we omit the technical proofs and move on with establishment of property 1. Pick any point on the horizontal axis $\boldsymbol{x} = (x, 0), x > 0$. By the proof of the utility representation in lecture and strong monotonicity, $\exists 0 < w < x$ such that $\boldsymbol{x} \sim \boldsymbol{w} = w\boldsymbol{e} = (w, w)$. Connect $\boldsymbol{x}$ and $\boldsymbol{w}$ and extend it to the vertical axis. Denote the intersection of the ray $\boldsymbol{xw}$ with the vertical axis as $\boldsymbol{y} = (0, y)$. Jointly from Lemma 1 and 2 we can say the points on the line $\boldsymbol{xy}$ are indifferent to each other. Finally, by strong monotonicity, the indifference curves must be downward sloping and for any $x \neq x'$, we cannot have $(x, 0) \sim (x', 0)$. Moreover, if $(x, 0) \sim (w, w)$, then for any $t \geq -w$, we have $(x + t, 0) \sim (w + t, w)$, so the indifference curves are parallel.

2.4 Expenditure Minimization Problem

To separate the substitution effect from the income effect of a price change, we introduce a dual problem: minimize expenditure subject to achieving a fixed utility level. This is the *Expenditure Minimization Problem* (EMP), and it will prove indispensable for the Slutsky decomposition later in this chapter.

Definition 2.4.1: Expenditure Minimization Problem

$$\begin{aligned} &\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \\ \text{s.t. } &u(\boldsymbol{x}) \geq u \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

Definition 2.4.2: Expenditure Function; Hicksian Demand Correspondence

Let $F(\boldsymbol{p}, u) = \{\boldsymbol{x} \in \mathbb{R}^n_+ : u(\boldsymbol{x}) \geq u\}$ be the feasible set. Define the optimal (minimal) value function as the *expenditure function*:

$$e(\boldsymbol{p}, u) = \inf_{\boldsymbol{x} \in F(\boldsymbol{p}, u)} \boldsymbol{p} \cdot \boldsymbol{x},$$

and the consumer's optimal choice(s) as the *Hicksian demand correspondence*:

$$\boldsymbol{x}^H(\boldsymbol{p}, u) = \{\boldsymbol{x} \in F(\boldsymbol{p}, u) : \boldsymbol{p} \cdot \boldsymbol{x} = e(\boldsymbol{p}, u)\}.$$

Because the utility level is held fixed, a price change leaves the consumer's welfare unchanged by construction. The resulting movement in Hicksian demand is therefore a pure ***substitution effect*** *— no income effect contaminates it.*

As we did for the UMP, we approach the EMP from two angles:

- When can we simplify the problem (e.g., existence of solution(s), binding utility level and uniqueness of solution)?
- Properties of expenditure function and Hicksian demand correspondence.

2.4.1 Existence of Solution

Proposition 2.4.3

Suppose $u(\cdot)$ represents a continuous preference relation and that $\boldsymbol{p} \gg \boldsymbol{0}$ and $F(u) \neq \varnothing$, then the expenditure minimization problem has at least one minimizer, i.e.,

$$\boldsymbol{x}^H(\boldsymbol{p}, u) \neq \varnothing.$$

Proof for Proposition.

- Pick any $\boldsymbol{x}_0 \in F(u)$ and consider the alternative feasible set:

$$\tilde{F} = \{\boldsymbol{x} \in \mathbb{R}^n_+ : u(\boldsymbol{x}) \geq u \text{ and } \boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{x}_0\}.$$

  It is easy to see that the expenditure minimization problem with the feasible set $\tilde{F}$ has the same solution as the original problem. Moreover, the key motivation for this construction is that $\tilde{F}$ is closed and bounded, and hence compact.

  - Boundedness is crafted by picking $\boldsymbol{x}_0$ purposefully but without loss of generality.
  - Closedness is guaranteed by continuity of $u(\cdot)$.
    - Note that $u(\cdot)$ need not be continuous, because we can always find an alternative continuous function $v(\cdot)$ that represents the same preference relation regardless of the continuity of $u(\cdot)$. Thus, $\tilde{F}$ is closed.
    - Note that the continuity of preference relation has no direct relation to the continuity of its utility representation.

- The objective function $\boldsymbol{p} \cdot \boldsymbol{x}$ is continuous. We know that a continuous function on a compact set has at least one minimizer, which ends our proof.

■

2.4.2 Binding Utility Level

Proposition 2.4.4

Suppose $u(\cdot)$ represents a continuous preference relation and that $\boldsymbol{p} \gg \mathbf{0}$, $u \geq u(\mathbf{0})$ and $F(u) \neq \varnothing$, then at any minimizer $\boldsymbol{x}^*$,

$$u(\boldsymbol{x}^*) = u.$$

Proof for Proposition.

- By the argument in the previous proposition, we can assume without loss that $u(\cdot)$ is continuous. Suppose to the contrary that a minimizer $\boldsymbol{x}^*$, we have $u(\boldsymbol{x}^*) > u$. Since $u \geq u(\mathbf{0})$, $\boldsymbol{x}^* \neq \mathbf{0}$.
- By the continuity of $u(\cdot)$, we know for some $\varepsilon > 0$. $u\left((1-\varepsilon)\boldsymbol{x}^*\right) > u$. It follows that $(1-\varepsilon)\boldsymbol{x}^* \in F(u)$ and that $\boldsymbol{p} \cdot (1-\varepsilon)\boldsymbol{x}^* < \boldsymbol{p} \cdot \boldsymbol{x}^*$, which is a contradiction to the optimality of $\boldsymbol{x}^*$.

∎

Remark.

The binding condition here in EMP is much weaker than that in UMP, where we do not put "any" additional condition on preference relation. One can understand this as the objective function in EMP is itself locally non-satiated.

2.4.3 Unique Minimizer

Proposition 2.4.5

Suppose $u(\cdot)$ represents a continuous and strictly convex preference relation and that $\boldsymbol{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$. Then the expenditure minimization problem has exactly one minimizer, i.e., $\boldsymbol{x}^H(\boldsymbol{p}, u)$ is a singleton.

Proof for Proposition.

- By the preceding proposition, *at least one minimizer exists.*
- Suppose to the contrary that there are two minimizers $\boldsymbol{x}^* \neq \boldsymbol{y}^*$. Then by feasibility, $u(\boldsymbol{x}^*) \geq u$ and $u(\boldsymbol{y}^*) \geq u$. Since the preference relation is strictly convex, $u(\cdot)$ is strictly quasi-concave, so for the bundle $\mathbf{w} = \frac{1}{2}\boldsymbol{x}^* + \frac{1}{2}\boldsymbol{y}^* \neq \mathbf{0}$, we have $u(\mathbf{w}) > \min\{u(\boldsymbol{x}^*), u(\boldsymbol{y}^*)\} \geq u$. By continuity, for some small $\varepsilon > 0$, $u((1-\varepsilon)\mathbf{w}) \geq u$. Moreover, $\boldsymbol{p} \cdot (1-\varepsilon)\mathbf{w} = \frac{1}{2}(1-\varepsilon)\boldsymbol{p} \cdot \boldsymbol{x}^* + \frac{1}{2}(1-\varepsilon)\boldsymbol{p} \cdot \boldsymbol{y}^* < \boldsymbol{p} \cdot \boldsymbol{x}^* = \boldsymbol{p} \cdot \boldsymbol{y}^*$, which is a contradiction.

∎

2.4.4 Summary of Properties

Proposition 2.4.6: Properties of Expenditure Function and Hicksian Demand Correspondence

Suppose $\boldsymbol{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$.

- **Existence of minimizer**: If $u(\cdot)$ represents a continuous preference relation and that, then the Hicksian demand $\boldsymbol{x}^H(\boldsymbol{p}, u) \neq \varnothing$.
- **Structure of Hicksian demand**: If $u(\cdot)$ represents a convex preference relation, then $\boldsymbol{x}^H(\boldsymbol{p}, u)$ is a convex set. If $u(\cdot)$ represents a continuous and strictly convex preference relation, then $\boldsymbol{x}^H(\boldsymbol{p}, u)$ is a singleton.
- **Homogeneity**: $e(\boldsymbol{p}, u)$ is homogeneous of degree 1 in $\boldsymbol{p}$, that is, for any $t > 0$, $e(t\boldsymbol{p}, u) = te(\boldsymbol{p}, u)$.
- **Monotonicity of $e(\boldsymbol{p}, u)$**: $e(\boldsymbol{p}, u)$ is non-decreasing in $\boldsymbol{p}$ and u. If $u(\cdot)$ represents a continuous preference relation, then $e(\boldsymbol{p}, u)$ is strictly increasing in u when $u \geq u(\mathbf{0})$.
- **Binding utility level**: Suppose $u(\cdot)$ represents a continuous preference relation and $u \geq u(\mathbf{0})$. Then at any minimizer $\boldsymbol{x}^*$, $u(\boldsymbol{x}^*) = u$.

Since $\min \boldsymbol{p} \cdot \boldsymbol{x}$ is equivalent to $\max -\boldsymbol{p} \cdot \boldsymbol{x}$, EMP can be solved in an analogous manner to UMP. EMP and UMP share the same "tangency condition" for interior solutions:

$$\frac{MU_i}{MU_j} = \frac{p_i}{p_j}$$

Example.

Suppose a consumer's preference relation can be represented by the following utility function:

$$u(x_1, x_2) = \ln x_1 + x_2.$$

Moreover, the consumer faces constant prices $(p_1, p_2) \gg \mathbf{0}$ and has income $m > 0$.

1. Solve the consumer's utility maximization problem to derive the Marshallian demand $\boldsymbol{x}^M(p_1, p_2, m)$ and indirect utility function $v(p_1, p_2, m)$.
2. Solve the consumer's expenditure minimization problem to derive the Hicksian demand $\boldsymbol{x}^H(p_1, p_2, u)$ and expenditure function $e(p_1, p_2, u)$.

Solution.

1. It is easy to check that the preference relation is monotonic and strictly convex. The utility miximization can then be simplified as:

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) = \ln x_1 + x_2 \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m.$$

The "tangency condition" is: $|MRS| = \frac{MU_1}{MU_2} = \frac{1}{x_1} = \frac{p_1}{p_2}$.

Together with the binding budget constraint, we have $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = \frac{m}{p_2} - 1$.

Note that the constraint $x_2 \geq 0$ may be binding. Marshallian demand is given by

$$\boldsymbol{x}^M(p_1, p_2, m) = \begin{cases} \left(\frac{p_2}{p_1}, \frac{m}{p_2} - 1\right), \text{ if } m \geq p_2 \\ \left(\frac{m}{p_1}, 0\right), \text{ if } 0 < m < p_2 \end{cases}$$

2. Similar to the previous part, the expenditure minimization problem can be simplified as:

$$\min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2 \quad \text{s.t. } u(x_1, x_2) = \ln x_1 + x_2 = u.$$

The "tangency condition" is $|MRS| = \frac{MU_1}{MU_2} = \frac{1}{x_1} = \frac{p_1}{p_2}$.

Together with binding utility level, we have $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = u - \ln \frac{p_2}{p_1}$.

Notice the constraint $x_2 \geq 0$ may be binding, so the Hicksian demand is given by:

$$\boldsymbol{x}^H(p_1, p_2, m) = \begin{cases} \left(\frac{p_2}{p_1}, u - \ln \frac{p_2}{p_1}\right), \text{ if } u \geq \ln \frac{p_2}{p_1} \\ (e^u, 0), \text{ if } u < \ln \frac{p_2}{p_1} \end{cases}$$

Remember to check if the utility function is monotone and quasi-concave beforehand!

2.5 Duality and Comparative Statics

2.5.1 Duality between UMP and EMP

Recall consumer's utility maximization problem (UMP):

$$\begin{aligned} &\max_{\boldsymbol{x}} u(\boldsymbol{x}) \\ \text{s.t. } &\boldsymbol{p} \cdot \boldsymbol{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

Consumer's expenditure minimization problem (EMP):

$$\begin{aligned} &\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \\ \text{s.t. } &u(\boldsymbol{x}) \geq u \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

The two problems swap the roles of objective and constraint: the UMP's objective $u(\boldsymbol{x})$ becomes the EMP's constraint, and the UMP's constraint $\boldsymbol{p} \cdot \boldsymbol{x}$ becomes the EMP's objective. In optimization language they are *dual problems*, and the duality forces a tight relationship between Marshallian and Hicksian demand.

Proposition 2.5.1

Suppose $u(\cdot)$ is a utility function that represents a continuous and locally non-satiated preference relation on $X = \mathbb{R}^n_+$, then for any $\boldsymbol{p} \gg \mathbf{0}$, we have:

1. For any $m \geq 0$, $\boldsymbol{x}^M(\boldsymbol{p}, m) = \boldsymbol{x}^H(\boldsymbol{p}, v(\boldsymbol{p}, m))$ and $e(\boldsymbol{p}, v(\boldsymbol{p}, m)) = m$.
2. For any $u \geq u(\mathbf{0})$, $\boldsymbol{x}^H(\boldsymbol{p}, u) = \boldsymbol{x}^M(\boldsymbol{p}, e(\boldsymbol{p}, u))$ and $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$.

Proof for Proposition.

- Fix $m > 0$ and any $\boldsymbol{x}_0^M \in \boldsymbol{x}^M(\boldsymbol{p}, m)$, we have:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, m)) \leq \boldsymbol{p} \cdot \boldsymbol{x}_0^M = m.$$

  Fix any $u \geq u(\mathbf{0})$ and any $\boldsymbol{x}_0^H \in \boldsymbol{x}^H(\boldsymbol{p}, u)$, we have

$$v(\boldsymbol{p}, e(\boldsymbol{p}, u)) \geq u(\boldsymbol{x}_0^H) = u.$$

- Applying the first inequality to the wealth level $m = e(\boldsymbol{p}, u)$, we have:

$$e(\boldsymbol{p}, v(\boldsymbol{p}, e(\boldsymbol{p}, u))) \leq e(\boldsymbol{p}, u).$$

  On the other hand, since the preference relation is continuous, $e(\boldsymbol{p}, u)$ is strictly increasing in u, so from the second inequality, we have

$$e(\boldsymbol{p}, v(\boldsymbol{p}, e(\boldsymbol{p}, u))) \geq e(\boldsymbol{p}, u).$$

- Again by strict monotonicity of $e(\boldsymbol{p}, u)$ in u, we have $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$. Similarly, $e(\boldsymbol{p}, v(\boldsymbol{p}, m)) = m$.

- Finally, since the preference relation is continuous and locally non-satiated, the budget constraint must bind at the UMP and the utility level must bind at the EMP. Correspondingly,

$$\boldsymbol{x}^M(\boldsymbol{p}, m) = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{p} \cdot \boldsymbol{x} = m, u(\boldsymbol{x}) = v(\boldsymbol{p}, m)\} = \boldsymbol{x}^H(\boldsymbol{p}, v(\boldsymbol{p}, m))$$
$$\boldsymbol{x}^H(\boldsymbol{p}, u) = \{\boldsymbol{x} \in \mathbb{R}^n_+ : u(\boldsymbol{x}) = u, \boldsymbol{p} \cdot \boldsymbol{x} = e(\boldsymbol{p}, u)\} = \boldsymbol{x}^M(\boldsymbol{p}, e(\boldsymbol{p}, u))$$

Intuitively: pick a wealth level m and let $u = v(\boldsymbol{p}, m)$ be the utility the consumer actually attains; then the Marshallian demand at $(\boldsymbol{p}, m)$ and the Hicksian demand at $(\boldsymbol{p}, u)$ pick out the same bundles. The dual problems describe the same optimum from two angles.

2.5.2 Envelope Theorem

Theorem 2.5.2: Envelope Theorem for Unconstrained Optimization

Let $f : X \times [\underline{\theta}, \bar{\theta}] \to \mathbb{R}$ and $V(\theta) = \sup_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \theta)$. Suppose $f(\boldsymbol{x}, \cdot)$ is differentiable in θ for all $\boldsymbol{x} \in X$. Moreover, there exists an integrable function $b : [\underline{\theta}, \bar{\theta}] \to \mathbb{R}_+$ such that $|f_\theta(\boldsymbol{x}, \theta)| \leq b(\theta)$ for all $\boldsymbol{x} \in X$ and almost all $\theta \in [\underline{\theta}, \bar{\theta}]$. Then $V(\cdot)$ is absolutely continuous and hence differentiable almost everywhere. In addition, for any $\boldsymbol{x}^* \in \arg\max_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \theta)$,

$$V'(\theta) = f_2(\boldsymbol{x}^*(\theta), \theta).$$

Remark.

- The intuition is that, only the direct effect matters; the indirect effect of $\boldsymbol{x}^*(\theta)$ on $V(\theta)$ can be ignored, since $f(\cdot)$ does not change with $\boldsymbol{x}$ at the optimum.
- May get some inspiration from the simplest version.
  - Suppose x is one-dimensional and that the optimizer is unique and differentiable in θ. Combined with F.O.C., we would have:

    $$V'(\theta) = f_1(x^*(\theta), \theta) \cdot (x^*)'(\theta) + f_2(x^*(\theta), \theta) = f_2(x^*(\theta), \theta).$$

  - For the envelope theorem to hold, $\boldsymbol{x}$ need not be one-dimensional, $f(\cdot, \theta)$ need not be differentiable in $\boldsymbol{x}$, and the optimal $\boldsymbol{x}^*(\theta)$ need not be unique or differentiable in θ.

Theorem 2.5.3: Envelope Theorem for Constrained Optimization

Suppose X is compact and convex. Let $f, g : X \times \left[\underline{\theta}, \bar{\theta}\right] \to \mathbb{R}$ and $V(\theta) = \sup_{\boldsymbol{x} \in X : g(\boldsymbol{x}, \theta) \geq 0} f(\boldsymbol{x}, \theta)$. The Lagrangian is given by $\mathcal{L}(\boldsymbol{x}, \theta; \lambda) = f(\boldsymbol{x}, \theta) + \lambda g(\boldsymbol{x}, \theta)$. Suppose f and g are continuous and concave in $\boldsymbol{x}$, $f_2(\boldsymbol{x}, \theta)$ and $g_2(\boldsymbol{x}, \theta)$ are continuous in $(\boldsymbol{x}, \theta)$, and there exists $\boldsymbol{x}_0 \in X$ such that $g(\boldsymbol{x}_0, \theta) > 0$ for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$. Then $V(\cdot)$ is absolutely continuous and hence differentiable (a.e.). In addition, for any $\boldsymbol{x}^*(\theta) \in \arg\max_{\boldsymbol{x} \in X : g(\boldsymbol{x}, \theta) \geq 0} f(\boldsymbol{x}, \theta)$,

$$V'(\theta) = \mathcal{L}_2(\boldsymbol{x}^*(\theta), \theta; \lambda^*).$$

Roy's identity and Shephard's lemma are the two canonical applications of the envelope theorem in consumer theory — one for the UMP, one for the EMP.

Corollary 2.5.4: Roy's Identity

Suppose $u(\cdot)$ represents a locally non-satiated and strictly convex preference relation on $X = \mathbb{R}^n_+$. Then, for any $(\boldsymbol{p}, m) \gg \boldsymbol{0}$, the Marshallian demand for good i, $x_i^M(\boldsymbol{p}, m)$, is given by:

$$x_i^M(\boldsymbol{p}, m) = -\frac{\partial v(\boldsymbol{p}, m)/\partial p_i}{\partial v(\boldsymbol{p}, m)/\partial m}.$$

Proof for Corollary.

The Lagrangian of the utility maximization problem is given by:

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu; \boldsymbol{p}, m) = u(x_1, x_2, \cdots, x_n) + \lambda \left(m - \sum_{i=1}^n p_i x_i \right) + \sum_{i=1}^n \mu_i x_i$$

By the envelope theorem, we have:

$$\begin{cases} \frac{\partial v(\boldsymbol{p}, m)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}|_{(\boldsymbol{x}^*, \lambda^*, \mu^*)} = -\lambda^* x_i^M(\boldsymbol{p}, m) \\ \frac{\partial v(\boldsymbol{p}, m)}{\partial m} = \frac{\partial \mathcal{L}}{\partial m}|_{(\boldsymbol{x}^*, \lambda^*, \mu^*)} = \lambda^* \end{cases} \implies x_i^M(\boldsymbol{p}, m) = -\frac{\partial v(\boldsymbol{p}, m)/\partial p_i}{\partial v(\boldsymbol{p}, m)/\partial m}$$

Corollary 2.5.5: Shephard's Lemma

Suppose $u(\cdot)$ represents a locally non-satiated and strictly convex preference relation on $X = \mathbb{R}^n_+$. Then, for any $\boldsymbol{p} \gg \boldsymbol{0}$ and $F(u) \neq \varnothing$, the Hicksian demand for good i, $x_i^H(\boldsymbol{p}, u)$, is given by:

$$x_i^H(\boldsymbol{p}, u) = \frac{\partial e(\boldsymbol{p}, u)}{\partial p_i}.$$

Proof for Corollary.

The Lagrangian of the expenditure minimization problem is given by:

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu; \boldsymbol{p}, u) = \sum_{i=1}^n p_i x_i - \lambda(u(\boldsymbol{x}) - u) - \sum_{i=1}^n \mu_i x_i.$$

By the envelope theorem, we have:

$$\frac{\partial e(\boldsymbol{p}, u)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}|_{(\boldsymbol{x}^*, \lambda^*, \mu^*)} = x_i^H(\boldsymbol{p}, u).$$

By strict convexity, both Marshallian demand and Hicksian demand are single-valued.

2.5.3 Slutsky Equation

The Slutsky equation is the formal decomposition we have been building toward: it splits the Marshallian response to a price change into a substitution effect (the Hicksian piece) and an income effect.

Theorem 2.5.6: Slutsky Equation

Suppose $u(\cdot)$ represents a continuous, locally non-satiated and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^n_+$ and that $\boldsymbol{x}^M(\boldsymbol{p}, m)$ and $\boldsymbol{x}^H(p, u)$ are both differentiable and single-valued. Then

$$\frac{\partial x_i^M(\boldsymbol{p}, m)}{\partial p_j} = \frac{\partial x_i^H(\boldsymbol{p}, v(\boldsymbol{p}, m))}{\partial p_j} - \frac{\partial x_i^M(\boldsymbol{p}, m)}{\partial m} x_j^M(\boldsymbol{p}, m).$$

Proof for Theorem

The proof proceeds with duality of Marshallian and Hicksian demand throughout.

$$x_i^H(\boldsymbol{p}, u) = x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, u)), \text{ as long as } u \geq u(\mathbf{0})$$

Take partial derivatives with respect to p_j:

$$\frac{\partial x_i^H(\boldsymbol{p}, u)}{\partial p_j} = \frac{x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, u))}{\partial p_j} + \frac{\partial x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, u))}{\partial m} \cdot \frac{\partial e(\boldsymbol{p}, u)}{\partial p_j}$$

By Shephard's lemma:

$$\frac{\partial e(\boldsymbol{p}, u)}{\partial p_j} = x_j^H(\boldsymbol{p}, u)$$

By duality,

$$\begin{cases} x_j^H(\boldsymbol{p}, v(\boldsymbol{p}, m)) = x_j^M(\boldsymbol{p}, m) \\ e(\boldsymbol{p}, v(\boldsymbol{p}, m)) = m \end{cases}$$

Evaluating the partial derivative equation at $u = v(\boldsymbol{p}, m)$, we have:

$$\frac{\partial x_i^H(\boldsymbol{p}, v(\boldsymbol{p}, m))}{\partial p_j} = \frac{x_i^M(\boldsymbol{p}, e(\boldsymbol{p}, v(\boldsymbol{p}, m)))}{\partial p_j} + \frac{\partial x_i^M(\boldsymbol{p}, m)}{\partial m} \cdot x_j^M(\boldsymbol{p}, m)$$

Rearrange the terms to finish the proof of Slutsky Equation. ■

The economic content: the LHS is the total Marshallian response of good i's demand to a change in p_j; the first term on the RHS is the substitution effect (how the consumer would re-allocate at a fixed utility level), and the second term is the income effect (the price change effectively makes the consumer richer or poorer at the original bundle, by an amount proportional to her current consumption of good j).

The Slutsky equation seeds a battery of comparative-statics classifications. We introduce them next.

Comparative statics in income. A normal good is one for which demand rises with wealth; an inferior good is one for which demand falls.

Definition 2.5.7: Normal Good; Inferior Good

Good i is a *normal good* if $x_i^M(\boldsymbol{p}, m)$ is increasing in m, an *inferior good* if $x_i^M(\boldsymbol{p}, m)$ is decreasing in m.

"Inferior" is a technical label about how demand co-moves with income; it carries no claim about the good's quality.

Comparative statics in own price. The next pair captures how Marshallian demand responds to a change in the good's own price.

Definition 2.5.8: Regular Good; Giffen Good

Good i is a *regular good* if $x_i^M(\boldsymbol{p}, m)$ is decreasing in p_i, a *Giffen good* if $x_i^M(\boldsymbol{p}, m)$ is increasing in p_i.

A Giffen good must be inferior. The substitution effect of a price increase always pushes demand down (the Hicksian substitution matrix is negative semi-definite); for total demand to rise *with the price, the income effect must be both positive in absolute value* and *large enough to overwhelm the substitution effect — which requires the good to be inferior.*

Comparative statics in cross-price. The last pair captures how the demand for good i responds to a price change in good j.

Definition 2.5.9: Substitute; Complement

Good i is a *substitute* for good j if $x_i^H(\boldsymbol{p}, u)$ is increasing in p_j, a *complement* for good j if $x_i^H(\boldsymbol{p}, u)$ is decreasing in p_j.

Good i being a complement for good j means that, any increase in p_j would shift part of the original share of consumption onto alternative goods other than good i and good j.

Definition 2.5.10: Gross Substitute; Gross Complement

Good i is a *gross substitute* for good j if $x_i^M(\boldsymbol{p}, m)$ is increasing in p_j, a *gross complement* for good j if $x_i^M(\boldsymbol{p}, m)$ is decreasing in p_j.

2.6 Consumer Welfare

The textbook measure of consumer welfare is consumer surplus — the area between the demand curve and the price line. But it has two limitations:

- It is a partial-equilibrium tool: it does not gracefully handle multiple simultaneous price changes.
- It has no clean interpretation in terms of utility itself.

We supplement it with two utility-based welfare measures — *compensating variation* and *equivalent variation* — both expressed in dollar units of "equivalent wealth."

Definition 2.6.1: Compensating Variation; Equivalent Variation

Suppose the initial price is $\boldsymbol{p}^0$ and $u^0 = v(\boldsymbol{p}, m)$, and that the final price is $\boldsymbol{p}'$ and $u' = v(\boldsymbol{p}', m)$. Compensating variation and equivalent variation are defined as:

1. *Compensating variation*: $CV = e(\boldsymbol{p}^0, u^0) - e(\boldsymbol{p}', u^0)$.
2. *Equivalent variation*: $EV = e(\boldsymbol{p}^0, u') - e(\boldsymbol{p}', u')$.

Notice that CV and EV have the same sign, and is positive for a price drop and negative for a price increase (though the two cases are not exhaustive).

Mathematically, by duality,

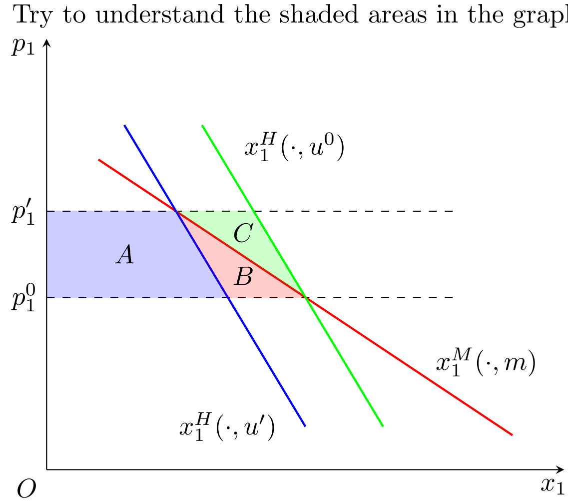
$$\begin{aligned} CV &= e(\boldsymbol{p}^0, u^0) - e(\boldsymbol{p}', u^0) = m - e(\boldsymbol{p}', u^0) \\ EV &= e(\boldsymbol{p}^0, u') - e(\boldsymbol{p}', u') = e(\boldsymbol{p}^0, u') - m \end{aligned}$$

Intuitively, $-CV$ measures how much we need to *compensate* the consumer for them to achieve the original level of utility at the new price vector, while EV measures what is the equivalent amount of money that the consumer values this price change if the price vector were fixed at the original level.

Suppose the price of a single good i changes from p_i^0 to p_i', then

$$\begin{aligned} CV &= \int_{p_i'}^{p_i^0} \frac{\partial e(\boldsymbol{p}, u^0)}{\partial p_i} \mathrm{d}p_i = \int_{p_i'}^{p_i^0} x_i^H(\boldsymbol{p}, u^0) \mathrm{d}p_i \\ EV &= \int_{p_i'}^{p_i^0} \frac{\partial e(\boldsymbol{p}, u')}{\partial p_i} \mathrm{d}p_i = \int_{p_i'}^{p_i^0} x_i^H(\boldsymbol{p}, u') \mathrm{d}p_i \\ \Delta CS &= \int_{p_i'}^{p_i^0} x_i^M(\boldsymbol{p}, m) \mathrm{d}p_i \end{aligned}$$

Suppose the price of good 1 increases from p_1^0 to p_1'. The change is depicted in the following graph, with Marshallian demand, Hicksian demands and the change of CV, EV and ΔCS. Try to understand the shaded areas in the graph.



$$\begin{cases} -CV = A + B + C \\ -EV = A \\ -\Delta CS = A + B \end{cases}$$

Remark.

1. If the Marshallian demand curve is steeper than the Hicksian demand curve, it implies that the good is an inferior good. In the graph, the represented good is a normal good.

2. On any range where the good in question is either normal or inferior, then:

$$\min\{CV, EV\} \leq \Delta CS \leq \max\{CV, EV\}.$$

   Notice that the two cases are not exhaustive. For example, a good may be normal good at some lower ranger of price, but reversed to inferior good at higher range of price.

Example.

Suppose a consumer has a locally non-satiated and strictly convex preference relation on $\mathbb{R}^2_+$ that can be represented by a twice continuously differentiable utility function $u(x_1, x_2) \geq 0$. Moreover, for $(p_1, p_2) \gg \mathbf{0}$ and $u \geq 0$, the expenditure function is given by:

$$e(p_1, p_2, u) = \frac{p_1 p_2 u^2}{p_1 + p_2}$$

1. For $(p_1, p_2) \gg \mathbf{0}$ and $u > 0$, derive the Hicksian demand $\boldsymbol{x}^H(p_1, p_2, u)$.

2. For $(p_1, p_2, m) \gg \mathbf{0}$, derive the Marshallian demand $\boldsymbol{x}^M(p_1, p_2, m)$.

3. Now suppose $p_2 = 1$ and $m = 2$. Consider a price drop from $p_1^0 = 2$ to $p_1' = 1$. Calculate the compensating variation (CV), the equivalent variation (EV), and the change in consumer surplus (ΔCS) of this price change.

Solution.

1. By Shephard's Lemma,

$$x_1^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{p_2^2 u^2}{(p_1 + p_2)^2}$$
$$x_2^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{p_1^2 u^2}{(p_1 + p_2)^2}$$

   It follows that $\boldsymbol{x}^H(p_1, p_2, u) = \left(\frac{p_2^2 u^2}{(p_1+p_2)^2}, \frac{p_1^2 u^2}{(p_1+p_2)^2}\right)$.

2. By duality,

$$e(p_1, p_2, v(\boldsymbol{p}, m)) = \frac{p_1 p_2 v(\boldsymbol{p}, m)^2}{p_1 + p_2} = m.$$

   We then have $v(\boldsymbol{p}, m)^2 = \frac{p_1+p_2}{p_1 p_2} m$.

   Again by duality,

$$\boldsymbol{x}^M(p_1, p_2, m) = \boldsymbol{x}^H(p_1, p_2, v(\boldsymbol{p}, m)) = \left(\frac{p_2 m}{p_1(p_1 + p_2)}, \frac{p_1 m}{p_2(p_1 + p_2)}\right).$$

3. Apply the definition of CV, EV and ΔCS:

$$\begin{cases} CV = e(p_1^0, p_2, u^0) - e(p_1', p_2, u') = m - e(p_1', p_2, u^0) \\ EV = e(p_1^0, p_2, u') - e(p_1^0, p_2, u') = e(p_1^0, p_2, u') - m \\ \Delta CS = \int_1^2 x_1^M(p_1, p_2, m)\mathrm{d}p_1 = 2\ln\frac{4}{3} = 0.58 \end{cases}$$

Notice that $u^0 = v(p_1^0, p_2, m) = \sqrt{3}$ and $u' = v(p_1', p_2, m) = 2$. Then the results are:

$$\begin{cases} CV = \frac{1}{2} \\ EV = \frac{2}{3} \\ \Delta CS = 2\ln\frac{4}{3} \approx 0.58 \end{cases}$$

Note that typically we have $\min\{CV, EV\} \leq \Delta CS \leq \max\{CV, EV\}$. This can be used to double check the "correctness" of your result.

There is one thing noteworthy about aggregation. While we have laid the foundation for individually analyzing consumer's utility maximization problem, it may fail if we directly make the aggregation.

Example.

Consider two consumers 1 and 2, whose preferences can be represented by the following utility functions:

$$u^1(x_1, x_2) = \begin{cases} x_1 x_2^3 & \text{if } 0 \leq x_2 \leq 7.7 \\ (7.7)^3 x_1 & \text{if } x_2 \geq 7.7 \end{cases}$$

$$u^2(x_1, x_2) = \begin{cases} x_1^3 x_2 & \text{if } x_1 \geq 3x_2 \\ \frac{1}{3}x_1^4 & \text{if } 0 \leq x_1 \leq 3x_2 \end{cases}$$

Consider the following budget sets:

- Budget set A: $p_1 = p_2 = 2$, $m = 20$.
- Budget set B: $p_1 = 3$, $p_2 = 1$, $m = 20$.

Intuitively, the failure of aggregation is due to *diverse income effects*. For instance, in the example above, the price change has a positive income effect on consumer 1, but a negative income effect on consumer 2. Aggregation is possible in the special case where all consumers have the same wealth effect, that is, $\frac{\partial \boldsymbol{x}^i}{\partial m^i} = \frac{\partial \boldsymbol{x}^j}{\partial m^j}$, for every two consumers i, j and p, m^i, m^j.