Chapter 3

Production Theory

Production theory parallels consumer theory, with firms in place of consumers and profit in place of utility. We will build the firm's problem in stages: first describe what the firm *can* do (the production set), then ask what the firm *wants* (profit maximization or, when output is given, cost minimization).

3.1 Setups

We begin with the standard assumptions that justify treating the firm as a profit-maximizer in a competitive market.

Assumption 3.1.1

1. *Perfect/complete* information: no uncertainty about input/output prices, production technology, etc.
2. *Perfectly competitive* input and output markets: firms are price-takers in both input and output markets.
3. Input/output prices are *linear*, justified by perfectly competitive markets.
4. Goods are perfectly *divisible*.
5. The technology is *exogenously given*.
6. The firm's managers are *perfectly controlled by the owners/shareholders.*

Remark.

1. Assumptions 1, 2, and 6 are what make "maximize profit" an unambiguous objective. Drop any of them and the objective itself becomes contestable:
   - Drop 1 (perfect information): owners with different risk preferences will disagree about which uncertain profit stream to maximize.
   - Drop 2 (price-taking): an owner who also has market power on the input or output side may prefer outcomes that distort the firm away from profit maximization.

- Drop 6 (perfect alignment): managers may pursue their own objectives — the agency problem.

2. Why is profit maximization determined by assumption 6 specifically? Consider a firm jointly owned by I consumers, with consumer i holding share $\theta_i \geq 0$ and $\sum_i \theta_i = 1$. Consumer i's utility maximization problem is

$$\max_{\boldsymbol{x}_i \geq \mathbf{0}} u_i(\boldsymbol{x}_i)$$
$$\text{s.t. } \boldsymbol{p} \cdot \boldsymbol{x}_i \leq m_i + \theta_i \boldsymbol{p} \cdot \boldsymbol{y}$$

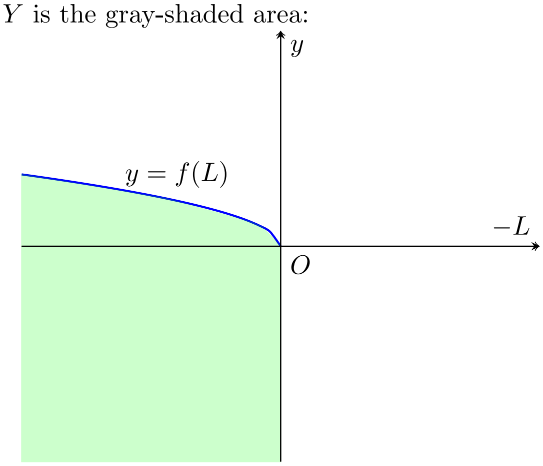
   Under local non-satiation, $v(\boldsymbol{p}, m)$ is strictly increasing in m. Every shareholder therefore strictly prefers higher $\boldsymbol{p} \cdot \boldsymbol{y}$ — the firm's profit — regardless of how heterogeneous their preferences over consumption bundles are. The unanimous shareholder objective is profit maximization. Drop assumption 6 (perfect manager-owner alignment) and managers might not implement this objective.

3.1.1 Production Set

Definition 3.1.2: Production Plan; Production Set

A *production plan* is a vector $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, where y_i can be either positive or negative, with $y_i > 0$ standing for an output and $y_i < 0$ an input.
The *production set* of a firm is described by $Y \subset \mathbb{R}^n$, where any $\boldsymbol{y} \in Y$ is a *feasible* production plan, i.e., a production plan the firm can choose from.

Example.
For example, consider the example of one input and one output, suppose the production set is given by $Y = \{(-L, y) : y \leq f(L), L \geq 0\}$, where $f(\cdot)$ is the production function. The production set Y is the gray-shaded area:



Throughout our analysis, we will make the innocent technical assumptions that Y is *non-empty* (so as to have something to study), *closed* (to help ensure the existence of optimal production plans), and $Y \neq \mathbb{R}^n$ (so that there is some scarcity). In addition to those, some other assumptions are needed to make the problem more practical.

Assumption 3.1.3: Production Set

1. $Y \neq \varnothing$.
2. Y is *closed*.
3. *No free lunch* and the possibility of *shutdown*: $Y \cap \mathbb{R}^n_+ = \{\mathbf{0}\}$.
4. *Free disposal*: $\boldsymbol{y} \in Y \implies \boldsymbol{y}' \in Y$, for any $\boldsymbol{y}' \leq \boldsymbol{y}$.

Remark.

Recall the distinction between the short run and the long run from intermediate microeconomics:

- Short run: some inputs are fixed.
- Long run: all inputs are variable.

In our discussion, we will mostly focus on the **long run** in advanced microeconomics. That is, all inputs are by default changeable.

3.1.2 Firm's Profit Maximization Problem

In the spirit of rational decision-making, the firm's problem can be framed as choosing the profit-maximizing production plan from its production set.

Definition 3.1.4: Profit Maximization Problem

$$\max_{\boldsymbol{y}} \boldsymbol{p} \cdot \boldsymbol{y}$$
$$\text{s.t. } \boldsymbol{y} \in Y$$

Similarly, we define the firm's profit function and optimal supply correspondence, following the same logic with consumer theory.

Definition 3.1.5: Profit Function

Profit function is defined as the optimal value function of the firm's profit given $\boldsymbol{p}$:

$$\pi(\boldsymbol{p}) = \sup_{y \in Y} \boldsymbol{p} \cdot \boldsymbol{y}.$$

Definition 3.1.6: Optimal Supply Correspondence

Optimal supply correspondence is defined as the firm's optimal choice(s) given $\boldsymbol{p}$:

$$\boldsymbol{y}^*(\boldsymbol{p}) = \arg\max_{\boldsymbol{y} \in Y} \boldsymbol{p} \cdot \boldsymbol{y} = \{\boldsymbol{y} \in Y : \boldsymbol{p} \cdot \boldsymbol{y} = \pi(\boldsymbol{p})\}.$$

Remark.

- We have not made sufficient assumptions to ensure that a maximum profit is achieved (i.e., $y^*(\boldsymbol{p}) \neq \varnothing$), so the sup in profit function cannot always be replaced with the max.

  Example.

  > A firm uses labor and capital to produce its sole output. The production function is given by $f(L, K) = \sqrt{LK}$. Suppose $p = 4$ and $w = r = 1$. If the firm chooses $L = K = t$, the profit is given by $\pi = 2t$, which is unbounded in t.

- $y^*(\boldsymbol{p})$, the optimal supply correspondence, is a set-valued function, which maps an element from one set, the domain of the function, to a subset of another set.

3.2 Profit Maximization and Rationalizability

The firm's analog of consumer theory's central questions is:

1. Given (some of) the firm's supply decisions $y(\boldsymbol{p})$ — but *not* the production set Y — when is $y(\cdot)$ consistent with profit maximization for some production set? (Rationalizability.)
2. When does the firm's profit maximization problem have a solution?
3. What properties do the profit function $\pi(\cdot)$ and the optimal supply correspondence $y^*(\cdot)$ inherit from the setup?
4. How do we actually solve the firm's problem?

These mirror the revealed-preference questions for consumers, but with the inference direction reversed. In consumer revealed preference we observe the feasible set and try to recover the objective; here we observe the objective (profits at various prices) and try to recover the feasible set (the production set).

In practice, we do not know a firm's production set Y, but observe some of its supply choice $y(\boldsymbol{p})$ for $\boldsymbol{p} \in \mathbb{R}^n$. Hence, we define rationalizability on empirical meanings.

Definition 3.2.1: Rationalizability

Empirical supply correspondence $y : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *rationalized* by production set Y if $y(\boldsymbol{p}) \subset y^*(\boldsymbol{p}) = \arg\max_{\boldsymbol{y} \in Y} \boldsymbol{p} \cdot \boldsymbol{y}$ for all $\boldsymbol{p} \in \mathbb{R}^n$. Empirical supply correspondence $y(\cdot)$ is *rationalizable* if it is rationalized by some production set.

Intuitively, an observed supply correspondence $y(\cdot)$ is rationalizable if we can find a production set Y such that $y(\cdot)$ is consistent with rational decision-making.

Remark.

- We may only observe some of the optimal choice(s) at each price $\boldsymbol{p}$, so in the definition it writes "$y(\boldsymbol{p}) \subset y^*(\boldsymbol{p})$".
- Practically, it is not necessary that we observe all of the firm's optimal supply decisions at all prices. We can define without loss that $y(\boldsymbol{p}_0) = \varnothing$ if the firm's supply decision

is not observed at $\boldsymbol{p}_0$.

We are naturally interested in what we can infer about the production set Y from the empirical observations if the supply choices are rationalizable. Suppose at price $\boldsymbol{p}$ the firm chooses production plan $y(\boldsymbol{p})$. Here are two plausible inferences:

1. Plan $y(\boldsymbol{p})$ must be feasible, i.e., $y(\boldsymbol{p}) \in Y$.
2. Any production plan $\boldsymbol{y}$ other than elements in $\boldsymbol{y}(\boldsymbol{p})$ must generate no more profits than elements in $y(\boldsymbol{p})$ at price $\boldsymbol{p}$. Or equivalently, any production plan $\boldsymbol{y}$ that is more profitable than $\boldsymbol{y}(\boldsymbol{p})$ at price $\boldsymbol{p}$ cannot be feasible.

We use the first idea to construct an "inner bound" on Y defined by all choices that the firm has actually made, as they must first be feasible to be chosen. We use the second idea to construct an "outer bound" on Y, which only includes plans that do not give the firm greater profits than its observed choices at any given price $\boldsymbol{p}$.

Definition 3.2.2: Inner Bound; Outer Bound

Given empirical supply correspondence $y(\cdot)$, we define the *inner bound* of the firm's production set as:

$$Y^I = \bigcup_{\boldsymbol{p} \in \mathbb{R}^n} y(\boldsymbol{p}),$$

and the *outer bound* of the firm's production set as:

$$Y^O = \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{y} \leq \boldsymbol{p} \cdot \boldsymbol{y}_{\boldsymbol{p}}, \forall \boldsymbol{p} \in \mathbb{R}^n \text{ and } \boldsymbol{y}_{\boldsymbol{p}} \in y(\boldsymbol{p})\}.$$

Intuitively, any optimal supply choice(s) must be feasible, so $Y^I \subset Y$; and any feasible production plan cannot yield a strictly higher profit at any price, so $Y \subset Y^O$. This intuition is formally characterized in the following proposition.

Proposition 3.2.3: Rationalizable Empirical Supply Correspondence

Production set Y rationalizes empirical supply correspondence $y(\cdot)$ if and only if

$$Y^I \subset Y \subset Y^O.$$

Proof for Proposition.

1. "Only if"
   - First consider any $\boldsymbol{z} \in Y^I$. By definition of Y^I, there exists a $\boldsymbol{p}$ such that $\boldsymbol{z} \in y(\boldsymbol{p})$. Since $y(\cdot)$ is rationalizable, $y(\boldsymbol{p}) \subset y^*(\boldsymbol{p}) \subset Y$. It follows that $\boldsymbol{z} \in Y$ and $Y^I \subset Y$.
   - Next consider any $\boldsymbol{y} \in Y$ and $\boldsymbol{p} \in \mathbb{R}^n$. Since $y(\cdot)$ is rationalizable, $y(\boldsymbol{p}) \subset y^*(\boldsymbol{p})$. By the definition of $y^*(\boldsymbol{p})$, for any $\boldsymbol{y}_{\boldsymbol{p}} \in y^*(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{y} \leq \boldsymbol{p} \cdot \boldsymbol{y}_{\boldsymbol{p}}$. It follows that $\boldsymbol{y} \in Y^O$ and $Y \subset Y^O$.
2. "If"
   - Fix $\boldsymbol{p} \in \mathbb{R}^n$ and consider any $\boldsymbol{y}_{\boldsymbol{p}} \in y(\boldsymbol{p})$.

- Since $y(\boldsymbol{p}) \subset Y^I \subset Y$, $\boldsymbol{y_p} \in Y$.
- Moreover, for any $\boldsymbol{y} \in Y \subset Y^O$, $\boldsymbol{p} \cdot \boldsymbol{y} \leq \boldsymbol{p} \cdot \boldsymbol{y_p}$. Consequently, $\boldsymbol{y_p} \in y^*(\boldsymbol{p})$.

Remark.

For the "if" part, by definition of Y^O, if we fix any $\boldsymbol{p} \in \mathbb{R}^n$, and take $\boldsymbol{y_p} \in y(\boldsymbol{p})$, $\boldsymbol{y_p}$ maximizes $\boldsymbol{p} \cdot \boldsymbol{y_p}$. However, with this condition we cannot simply conclude that $y(\cdot)$ is rationalizable, because $y_{\boldsymbol{p}}$ has to be in the production set Y, though it sounds trivial; or equivalently speaking, $\boldsymbol{y_p} \in Y^I$ may not fall in the outer bound Y^O.

The proposition indicates that, Y^I and Y^O carry all the information we have about the production set based on rational decision-making. The proposition immediately implies the following two corollaries:

Corollary 3.2.4: Weak Axiom of Profit Maximization (WAPM)

Let $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$. Empirical supply correspondence $y(\cdot)$ is rationalizable if and only if $Y^I \subset Y^O$, that is, $\boldsymbol{p} \cdot \boldsymbol{y_p} \geq \boldsymbol{p} \cdot \boldsymbol{y_{p'}}$, for any $\boldsymbol{p} \in P$ and $\boldsymbol{y_p} \in y(\boldsymbol{p})$, $\boldsymbol{y_{p'}} \in y(\boldsymbol{p}')$.

One simple consequence of this characterization is that, when checking rationalizability, we can restrict attention to supply functions rather than correspondences. (Simply put, compared with the preceding proposition, the production set Y is "left out" here.)

WAPM directly implies "law of supply".

Corollary 3.2.5: Law of Supply

Suppose empirical supply correspondence $y(\cdot)$ is rationalizable and let $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$. Then for any $\boldsymbol{p} \in P$ and $\boldsymbol{y_p} \in y(\boldsymbol{p})$, $\boldsymbol{y_{p'}} \in y(\boldsymbol{p}')$,

$$(\boldsymbol{p} - \boldsymbol{p}') \cdot (\boldsymbol{y_p} - \boldsymbol{y_{p'}}) \geq 0.$$

Proof for Corollary.

Since $y(\cdot)$ is rationalizable, by WAPM, we have

$$\boldsymbol{p} \cdot \boldsymbol{y_p} \geq \boldsymbol{p} \cdot \boldsymbol{y_{p'}}.$$

Switching the role of $\boldsymbol{p}$ and $\boldsymbol{p}'$, we have

$$\boldsymbol{p}' \cdot \boldsymbol{y_{p'}} \geq \boldsymbol{p}' \cdot \boldsymbol{y_p}$$

Adding the two equations above, we get the "law of supply".

In particular, if there is a single output and $y(\cdot)$ is single-valued, then

$$(p - p')(y(p) - y(p')) \geq 0$$

In other words, any rationalizable supply function must be **(weakly) upward sloping**.

Corollary 3.2.6

Let $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$. Empirical supply correspondence $y(\cdot)$ is rationalizable if and only if:

1. Any selection $\hat{y} : P \to \mathbb{R}^n$ is rationalizable.
2. For any two selections $\hat{y}$ and $\tilde{y}$ and any $\boldsymbol{p} \in P$, $\boldsymbol{p} \cdot \hat{y}(\boldsymbol{p}) = \boldsymbol{p} \cdot \tilde{y}(\boldsymbol{p})$. (Or equivalently, $\pi(\boldsymbol{p})$ is single-valued for each $\boldsymbol{p} \in P$.)

Remark.

- The first statement of this corollary is equivalent to WAPM applied to $\boldsymbol{p}' \neq \boldsymbol{p}$,
- The second statement of this corollary is equivalent to WAPM applied to $\boldsymbol{p}' = \boldsymbol{p}$.
- Thus, when given a supply correspondence, we only need to check that
  1. Each selection from it is a rationalizable supply function, and
  2. The profit function $\pi(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{y}$ is single-valued at any given $\boldsymbol{p} \in P$; or equivalently speaking, $\pi(\boldsymbol{p})$ does not depend on which selection is chosen.

  Since checking the second condition is trivial, we can focus on rationalizability of supply functions (single-valued correspondence).

Verifying rationalizability by checking all the WAPM inequalities is difficult when the set of observations is large. Fortunately, it turns out that when we have a continuum of observations, rationalizability can be verified much more easily using differential conditions. Specifically, we now suppose that we observe the firm's supply choices on an open convex set P of prices (e.g., P could be the set of all strictly positive price vectors).

Proposition 3.2.7: Rationalizability: Differentiable Case

Consider an empirical supply correspondence $y(\cdot)$ whose domain $P = \{\boldsymbol{p} \in \mathbb{R}^n : y(\boldsymbol{p}) \neq \varnothing\}$ is an open convex set. Suppose that $\pi(\boldsymbol{p}) = \boldsymbol{p} \cdot y(\boldsymbol{p})$ is a differentiable function on $\boldsymbol{p} \in P$. Then $y(\cdot)$ is rationalizable if and only if:

1. (**Hotelling's Lemma**) $\nabla \pi(\boldsymbol{p}) = \boldsymbol{y_p}$, for any $\boldsymbol{p} \in P$ and $\boldsymbol{y_p} \in y(\boldsymbol{p})$.
2. $\pi(\cdot)$ is a convex function.

Proof for Proposition.

1. "If"
   - Fix $\boldsymbol{q} \in P$ and take any $\boldsymbol{y_q} \in y(\boldsymbol{q})$. Consider the "difference function": $G(\boldsymbol{q}; \boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{y_q} - \pi(\boldsymbol{p})$.
   - It suffices to show that, $G(\boldsymbol{q}; \cdot)$ is maximized at $\boldsymbol{p} = \boldsymbol{q}$.
   - Since $\pi(\cdot)$ is a convex function, then $G(\boldsymbol{q}; \cdot)$ is a concave function. Since $G(\boldsymbol{q}; \cdot)$ is differentiable in $\boldsymbol{p}$, the first-order condition is both necessary and sufficient.

- The F.O.C.: $\boldsymbol{y_q} - \nabla\pi(\boldsymbol{p})|_{\boldsymbol{p}=\boldsymbol{q}} = 0$, which is precisely the Hotelling's Lemma.

2. "Only if"

- The proof above also shows WAPM implies the Hotelling's lemma. Indeed, it is just an application of envelope formula. It remains to show $\pi(\cdot)$ is a convex function.
- By rationalizability, there exists Y such that $\pi(\boldsymbol{p}) = \sup_{\boldsymbol{y}\in Y} \boldsymbol{p}\cdot\boldsymbol{y}$.
- Take any $\boldsymbol{p}, \boldsymbol{q} \in P$ and $t \in (0,1)$. If $\pi(\boldsymbol{p}) = +\infty$ or $\pi(\boldsymbol{q}) = +\infty$, then clearly $\pi(t\boldsymbol{p} + (1-t)\boldsymbol{q}) \le t\pi(\boldsymbol{p}) + (1-t)\pi(\boldsymbol{q})$. Otherwise,

$$\begin{aligned}\pi(t\boldsymbol{p} + (1-t)\boldsymbol{q}) &= \sup_{\boldsymbol{y}\in Y}(t\boldsymbol{p} + (1-t)\boldsymbol{q})\cdot\boldsymbol{y} \\ &\le t\cdot\sup_{\boldsymbol{y}\in Y}\boldsymbol{p}\cdot\boldsymbol{y} + (1-t)\cdot\sup_{\boldsymbol{y}\in Y}\boldsymbol{q}\cdot\boldsymbol{y} \\ &= t\pi(\boldsymbol{p}) + (1-t)\pi(\boldsymbol{q})\end{aligned}$$

Proposition 3.2.8: Rationalizability: General Case

Consider an empirical supply function $y : P \to \mathbb{R}^n$, where $P \subset \mathbb{R}^n$ is a convex set. $y(\cdot)$ is rationalizable if and only if:

1. (**Producer Surplus Formula**): $\pi(\boldsymbol{p}) = \boldsymbol{p}\cdot y(\boldsymbol{p})$ satisfies, for any smooth path $\rho : [0,1] \to P$, with $\rho(0) = \boldsymbol{p}$ and $\rho(1) = \boldsymbol{p}'$,

$$\pi(\boldsymbol{p}') - \pi(\boldsymbol{p}) = \int_0^1 y(\rho(t))\cdot\rho'(t)\mathrm{d}t.$$

2. (**Law of Supply**): For $\boldsymbol{p}, \boldsymbol{p}' \in P$,

$$(\boldsymbol{p} - \boldsymbol{p}')\cdot(y(\boldsymbol{p}) - y(\boldsymbol{p}')) \ge 0.$$

Proof for Proposition.

- "Only if" part

  We have already shown the "law of supply", so it suffices to show the "producer surplus formula".

  Let $\phi(t) = \pi(\rho(t))$. Consider the "difference function":

$$\begin{aligned}\delta(\theta; t) :=& \rho(t)\cdot y(\rho(\theta)) - \pi(\rho(t)) \\ =& \rho(t)\cdot y(\rho(\theta)) - \phi(t)\end{aligned}$$

By rationalizability of $y(\cdot)$,

$$\begin{aligned}
&\delta(\theta;t) \leq 0 = \delta(\theta;\theta) \\
\Longrightarrow &\left.\frac{\partial \delta(\theta;t)}{\partial t}\right|_{t=\theta} = 0 \\
\Longleftrightarrow &\left.\frac{\partial \delta(\theta;t)}{\partial t}\right|_{t=\theta} = y(\rho(\theta)) \cdot \rho'(\theta) - \phi'(\theta) = 0 \\
\Longleftrightarrow &\phi'(\theta) = y(\rho(\theta)) \cdot \rho'(\theta) \\
\Longrightarrow &\phi(1) - \phi(0) = \int_0^1 y(\rho(t)) \cdot \rho'(t)\mathrm{d}t \\
\Longrightarrow &\pi(\boldsymbol{p}') - \pi(\boldsymbol{p}) = \int_0^1 y(\rho(t)) \cdot \rho'(t)\mathrm{d}t
\end{aligned}$$

In fact in order to derive the integral result from the derivatives, we have to prove the continuity of $\phi(\cdot)$. It can be shown that $\phi(\cdot)$ is Lipschitz continuous and hence absolutely continuous. The proof is rather technical and thus omitted here.

- "If" part

In order to show the rationalizability of $y(\cdot)$, it suffices to show WAPM. Because the path integral is path-independent, take a straight line for math convenience:

$$\rho(t) = \boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p}).$$

We aim to prove

$$\pi(\boldsymbol{p}') - \boldsymbol{p}' \cdot y(\boldsymbol{p}) \geq 0.$$

Making tweaks to the difference in profit:

$$\begin{aligned}
\pi(\boldsymbol{p}') - \boldsymbol{p}' \cdot y(\boldsymbol{p}) &= \pi(\boldsymbol{p}') - \pi(\boldsymbol{p}) + \pi(\boldsymbol{p}) - \boldsymbol{p}' \cdot y(\boldsymbol{p}) \\
&= (\pi(\boldsymbol{p}') - \pi(\boldsymbol{p})) - (\boldsymbol{p}' - \boldsymbol{p}) \cdot y(\boldsymbol{p}) \\
&= \int_0^1 y(\rho(t)) \cdot \rho'(t)\mathrm{d}t - ((\boldsymbol{p}' - \boldsymbol{p}) \cdot y(\boldsymbol{p})) \\
&= \int_0^1 y(\boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p})) \cdot \rho'(t)\mathrm{d}t - ((\boldsymbol{p}' - \boldsymbol{p}) \cdot y(\boldsymbol{p})) \\
&= \int_0^1 (\boldsymbol{p}' - \boldsymbol{p}) \cdot \left[y\left(\boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p})\right) - y(\boldsymbol{p})\right]\mathrm{d}t
\end{aligned}$$

Let

$$\begin{cases} \boldsymbol{q}' = \boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p}) \\ \boldsymbol{q} = \boldsymbol{p} \end{cases}$$

One direct observation is that $\boldsymbol{q}' - \boldsymbol{q} = t(\boldsymbol{p}' - \boldsymbol{p})$. Therefore, we can simplify the preceding integral as:

$$\int_0^1 (\boldsymbol{p}' - \boldsymbol{p}) \cdot \left[y\left(\boldsymbol{p} + t(\boldsymbol{p}' - \boldsymbol{p})\right) - y(\boldsymbol{p})\right]\mathrm{d}t = \int_0^1 \frac{1}{t}(\boldsymbol{q}' - \boldsymbol{q}) \cdot (y(\boldsymbol{q}') - y(\boldsymbol{q}))\,\mathrm{d}t.$$

By law of supply, $(\boldsymbol{q}' - \boldsymbol{q}) \cdot (y(\boldsymbol{q}') - y(\boldsymbol{q})) \geq 0$. Therefore,

$$\int_0^1 \frac{1}{t}(\boldsymbol{q}' - \boldsymbol{q}) \cdot (y(\boldsymbol{q}') - y(\boldsymbol{q}))\, \mathrm{d}t \geq 0.$$

Example.

Consider a price-taking firm with m inputs and n outputs. Suppose we can observe the firm's input choices $\boldsymbol{z}$ for different input prices $\boldsymbol{w} \in \mathbb{R}_+^m$, but cannot observe its output choices or output prices (and may not even know the number of outputs n). We do know that output prices, whatever they are, do not change between the observations.

1. Suppose $m = 2$, and that at input prices $\boldsymbol{w}^1 = (1, 1)$ the firm chooses the input vector $\boldsymbol{z}^1 = (10, 15)$ and at input prices $\boldsymbol{w}^2 = (2, 3)$ it chooses the input vector $\boldsymbol{z}^2 = (13, 14)$. Is this pair of observations rationalizable (i.e., consistent with profit maximization for some production set and output prices)?

2. In general, give a necessary and sufficient condition for two input price-demand observations $\boldsymbol{z}^1, \boldsymbol{w}^1 \in \mathbb{R}_+^m$ and $\boldsymbol{z}^2, \boldsymbol{w}^2 \in \mathbb{R}^m$ to be rationalizable. Prove both necessity and sufficiency.

3. Now suppose instead that we have the following observations:

   - At prices $\boldsymbol{w}^1 = (1, 1)$, the firm chooses the input vector $\boldsymbol{z}^1 = (10, 15)$;
   - At prices $\boldsymbol{w}^2 = (2, 3)$, the firm chooses the input vector $\boldsymbol{z}^2 = (13, 13)$;
   - At prices $\boldsymbol{w}^3 = (4, 1)$, the firm chooses the input vector $\boldsymbol{z}^3 = (8, 9)$.

   Are these three observations jointly rationalizable?

Solution.

1. Suppose instead we know the output price $\boldsymbol{p}$ and output vectors $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, then by WAPM,

$$\begin{cases} (\boldsymbol{w}^1, \boldsymbol{p}) \cdot (-\boldsymbol{z}^1, \boldsymbol{y}^1) \geq (\boldsymbol{w}^1, \boldsymbol{p}) \cdot (-\boldsymbol{z}^2, y^2) \\ (\boldsymbol{w}^2, \boldsymbol{p}) \cdot (-\boldsymbol{z}^2, \boldsymbol{y}^2) \geq (\boldsymbol{w}^2, \boldsymbol{p}) \cdot (-\boldsymbol{z}^1, y^1) \end{cases}$$

$$\implies \begin{cases} \boldsymbol{p} \cdot \boldsymbol{y}^1 - 25 \geq \boldsymbol{p} \cdot \boldsymbol{y}^2 - 27 \\ \boldsymbol{p} \cdot \boldsymbol{y}^2 - 68 \geq \boldsymbol{p} \cdot \boldsymbol{y}^1 - 75 \end{cases}$$

$$\implies -2 \leq \boldsymbol{p} \cdot \boldsymbol{y}^1 - \boldsymbol{p} \cdot \boldsymbol{y}^2 \leq -3,$$

   which apparently leads to a contradiction.

2. In the same way, rationalizability requires that

$$\begin{aligned}&\begin{cases}(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,\boldsymbol{y}^1)\geq(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,y^2)\\(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,\boldsymbol{y}^2)\geq(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,y^1)\end{cases}\\\Longrightarrow&\boldsymbol{w}^1\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)\leq\boldsymbol{w}^2(\boldsymbol{z}^1-\boldsymbol{z}^2)\\\Longrightarrow&(\boldsymbol{w}^1-\boldsymbol{w}^2)\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq 0.\end{aligned}$$

So $(\boldsymbol{w}^1-\boldsymbol{w}^2)\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq 0$ is a necessary condition. Then prove it is also sufficient.

Take any output price $\boldsymbol{p}$ and output vectors $\boldsymbol{y}^1,\boldsymbol{y}^2$ that satisfies $\boldsymbol{w}^1\cdot(\boldsymbol{z}^1-\boldsymbol{z}^2)\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)\leq\boldsymbol{w}^2(\boldsymbol{z}^1-\boldsymbol{z}^2)$. It is trivial that output price $\boldsymbol{p}$ and production set $Y=\{(-z^1,\boldsymbol{y}^1),(-\boldsymbol{z}^2,\boldsymbol{y}^2)\}$ rationalize the pair of observations.

3. Again, when try to rationalize those choices, use necessary condition

$$\begin{cases}(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,\boldsymbol{y}^1)\geq(\boldsymbol{w}^1,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,y^2)\\(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^2,\boldsymbol{y}^2)\geq(\boldsymbol{w}^2,\boldsymbol{p})\cdot(-\boldsymbol{z}^1,y^1)\end{cases}$$

for pairwise checks. Then we can get

$$\begin{aligned}-1\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)&\leq 0\\22\leq\boldsymbol{p}\cdot(\boldsymbol{y}^2-\boldsymbol{y}^3)&\leq 24\\-14\leq\boldsymbol{p}\cdot(\boldsymbol{y}^3-\boldsymbol{y}^1)&\leq -8\end{aligned}$$

Even though at first glance there is no apparent contradiction, if we try to take the sum of the first two inequalities:

$$\begin{aligned}&\begin{cases}-1\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^2)\leq 0\\22\leq\boldsymbol{p}\cdot(\boldsymbol{y}^2-\boldsymbol{y}^3)\leq 24\end{cases}\\\Longrightarrow&21\leq\boldsymbol{p}\cdot(\boldsymbol{y}^1-\boldsymbol{y}^3)\leq 24\end{aligned}$$

which contradicts the third equation. Thus, the choices cannot be rationalized.

3.3 Profit Maximization Problem

We now turn from rationalizability to the firm's optimization problem itself. The questions parallel those for the consumer's UMP:

1. When does the profit maximization problem have a solution?
2. What properties does the profit function $\pi(\cdot)$ inherit, and what about the optimal supply correspondence $y^*(\cdot)$?
3. How do we solve the problem explicitly?

3.3.1 Returns to Scale

Recall the earlier counter-example where the optimal supply correspondence is empty. The pathology was that the firm could keep replicating its production plan and earn ever-greater profit — the problem has no maximum. To pin down when this happens we classify production sets by how they behave under scaling.

Definition 3.3.1: Returns to Scale

The production set Y exhibits:

- *non-increasing returns to scale* if $\boldsymbol{y} \in Y \implies t\boldsymbol{y} \in Y, \forall t \in [0,1]$.
- *non-decreasing returns to scale* if $\boldsymbol{y} \in Y \implies t\boldsymbol{y} \in Y, \forall t \in [1,+\infty)$.
- *constant returns to scale* if $\boldsymbol{y} \in Y \implies t\boldsymbol{y} \in Y, \forall t \geq 0$.

If the firm has non-decreasing returns to scale, there is no fundamental bound on its production capacity: scaling up the operation scales up profit. The result below makes this dichotomy precise — under non-decreasing returns, profits are either exactly zero or unbounded.

Proposition 3.3.2

If the production set $Y \neq \varnothing$ exhibits non-decreasing returns to scale, then for any $\boldsymbol{p} \in \mathbb{R}^n$, $\pi(\boldsymbol{p}) = 0$ or ∞.

Proof for Proposition.

First fix any $\boldsymbol{p} \in \mathbb{R}^n$. By the possibility of inaction or shutdown, $\boldsymbol{0} \in Y$, so $\pi(\boldsymbol{p}) \geq \boldsymbol{p} \cdot \boldsymbol{0} = 0$. Now suppose instead that $0 < \pi(\boldsymbol{p}) < \infty$, then there must exist $\boldsymbol{y}_0 \in Y$ such that $\boldsymbol{p} \cdot \boldsymbol{y}_0 > \pi(\boldsymbol{p}) - \varepsilon > 0$, for any $\varepsilon > 0$ small enough. By non-decreasing returns to scale, $t\boldsymbol{y} \in Y$, for any $t \geq 1$. Notice that $\boldsymbol{p} \cdot (t\boldsymbol{y}_0) = t(\boldsymbol{p} \cdot \boldsymbol{y}_0) > t\pi(\boldsymbol{p}) - t\varepsilon > \pi(\boldsymbol{p})$, for any $t > 1$ and $\varepsilon > 0$ small, which is a contradiction. It follows that $\pi(p) = 0$ or ∞. ∎

3.3.2 Properties of Profit Function and Supply Correspondence

Proposition 3.3.3: Properties of Profit Function and Supply Correspondence

Suppose the production set Y is closed and satisfies the free disposal property. Let $\pi(\cdot)$ be the profit function and $y^*(\cdot)$ the associated optimal supply correspondence. Then for $\boldsymbol{p} \gg \boldsymbol{0}$,

- $\pi(\cdot)$ is homogeneous of degree 1.
- $\pi(\cdot)$ is a convex function.
- $y^*(\cdot)$ is homogeneous of degree 0.
- If Y is a convex set, then $y^*(\boldsymbol{p})$ is a convex set for all $\boldsymbol{p} \gg \boldsymbol{0}$. If Y is a strictly convex set, then $y^*(\boldsymbol{p})$ is either empty or single-valued.
- (Hotelling's Lemma) If $y^*(\boldsymbol{p})$ is single-valued, then $\pi(\cdot)$ is differentiable at $\boldsymbol{p}$ and $\nabla\pi(\boldsymbol{p}) = y^*(\boldsymbol{p})$, that is,

$$\frac{\partial \pi(\boldsymbol{p}_i)}{\partial p_i} = y_i^*(\boldsymbol{p}), \text{ for } i = 1, 2, \cdots, n.$$

3.3.3 Derivation of Profit Maximization Problem

In preceding discussions, we seldom delve into the benchmark of judging if a production plan falls within the production set, i.e., being feasible. One convenient way to represent production possibility sets is using a transformation function $T : \mathbb{R}^n \to \mathbb{R}$, where $T(y) \leq 0$ implies that y is feasible, and $T(y) \geq 0$ implies that y is infeasible. The set of boundary points $\{y \in \mathbb{R}^n : T(y) = 0\}$ is called the transformation frontier.

Definition 3.3.4: Profit Maximization Problem

Suppose $T(\cdot)$ is the transformation function defining the production set:

$$\max_{\boldsymbol{y}} \boldsymbol{p} \cdot \boldsymbol{y}$$
$$\text{s.t. } T(\boldsymbol{y}) \leq 0$$

The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{y}, \lambda) = \boldsymbol{p} \cdot \boldsymbol{y} - \lambda T(\boldsymbol{y}) = \sum_{i=1}^{n} p_i y_i - \lambda T(\boldsymbol{y}).$$

The F.O.C.s are:

$$\lambda \cdot \nabla T(\boldsymbol{y}) = \boldsymbol{p}.$$

For most discussions in the course, we focus on single-output cases:

$$\max_{z} p \cdot f(\boldsymbol{z}) - \sum_{i=1}^{m} w_i z_i$$
$$\text{s.t. } \boldsymbol{z} \geq \mathbf{0}$$

In this special case, the Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{z}, \mu_i) = p \cdot f(\boldsymbol{z}) - \sum_{i=1}^{m} w_i z_i + \sum_{i=1}^{m} \mu_i z_i.$$

The F.O.C. are:

$$\begin{cases} p \cdot \frac{\partial f(\boldsymbol{z})}{\partial z_i} - w_i + \mu_i = 0 \\ \mu_i z_i = 0, \mu_i \geq 0 \end{cases}, \text{ for all } i = 1, 2, \cdots, m.$$

Equivalently, the F.O.C. can be written as

$$p \cdot \frac{\partial f(\boldsymbol{z})}{\partial z_i} \leq w_i, \text{ with equality if } z_i > 0.$$

Interpretation of the first-order conditions is straightforward and intuitive. The LHS represents the firm's marginal benefit from using an additional unit of input i, while the RHS represents the marginal cost. The F.O.C. require that the marginal benefit cannot exceed the marginal cost, with equality if the input is ever used at the optimum.

Similarly, we should follow systematic procedures to solve the firm's profit maximization problem:

- Check returns to scale and whether the production technology is strictly convex and monotonic.
- If decreasing returns to scale, production technology being monotonic and strictly convex, apply the "tangency conditions":

$$p \cdot MP_i = w_i, \forall i$$

- Check whether the inputs are non-negative, whether the profit is non-negative, and whether the profit is bounded.

Example.

A firm uses two inputs, labor (L) and capital (K), to produce a single output (Y). The production function is given by:

$$f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{2}}$$

Suppose the input and output prices are $(w, r, p) \gg \mathbf{0}$. Solve the firm's profit maximization problem to derive the profit function $\pi(w, r, p)$ and optimal supply correspondence $y^*(w, r, p)$.

Solution.

- Step 1: The production function is Cobb-Douglas, and hence strictly convex and monotonic. Moreover, if $f(L,K) \geq y$, then $f(tL,tK) = t^{\frac{3}{4}} f(L,K) \geq t(L,K) \geq ty$, for any $0 \leq t \leq 1$, so decreasing returns to scale.

- Step 2: F.O.C. are given by

$$\begin{cases} [L] : p \cdot \frac{\partial f(L,K)}{\partial L} = w \\ [K] : p \cdot \frac{\partial f(L,K)}{\partial K} = r \end{cases} \implies \begin{cases} L^* = \frac{p^4}{64w^2r^2} \\ K^* = \frac{p^4}{32wr^3} \end{cases}$$

- Check non-negativity:

  Clearly, $L^*, K^* > 0$. It follows that

$$\begin{cases} y^*(w,r,p) = \left(-\frac{p^4}{64w^2r^2}, -\frac{p^4}{32wr^3}, \frac{p^3}{16wr^2}\right) \\ \pi(w,r,p) = \frac{p^4}{64wr^2} > 0 \end{cases}$$

3.4 Cost Minimization Problem

Although the PMP solves the firm's problem directly, it is worthwhile to detour through the cost minimization problem (CMP) for two reasons:

- The CMP is better-behaved than the PMP — much like the EMP relative to the UMP in consumer theory, existence and uniqueness conditions are milder.
- When the firm has monopoly power in the output market but is still a price-taker in inputs, the indirect approach (first minimize cost for each output level, then optimize output) is the only tractable way to solve the problem.

3.4.1 Setups and Properties

Definition 3.4.1: Cost Function; Conditional Factor Demand Correspondence

Let $Z(y) = \{\boldsymbol{z} \in \mathbb{R}^n_+ : f(\boldsymbol{z}) \geq y\}$ be the firm's feasible set. We define the optimal (minimal) value function as the *cost function*:

$$c(\boldsymbol{w}, y) = \inf_{\boldsymbol{z} \in Z(y)} \boldsymbol{w} \cdot \boldsymbol{z},$$

and the firm's optimal factor choice(s) as the *conditional factor demand correspondence*:

$$\boldsymbol{z}(\boldsymbol{w}, y) = \{\boldsymbol{z} \in Z(y) : \boldsymbol{w} \cdot \boldsymbol{z} = c(\boldsymbol{w}, y)\}.$$

Notice that min $f(\boldsymbol{x})$ is equivalent to $\max(-f(\boldsymbol{x}))$, so the cost minimization problem can be viewed as profit maximization problem on the restricted production set

$$Y_y = \{(-\boldsymbol{z}, y) : \boldsymbol{z} \in \mathbb{R}^n_+, y \leq f(\boldsymbol{z})\}.$$

From this near-equivalence between the CMP and the PMP, the differentiable-case rationalizability result transfers immediately.

Proposition 3.4.2

Consider a conditional factor demand function $\boldsymbol{z} : W \times \mathbb{R} \to \mathbb{R}^m$ for a fixed output y on an open convex set $W \subset \mathbb{R}^m_+$ such that $c(\boldsymbol{w}, y) = \boldsymbol{w} \cdot \boldsymbol{z}(\boldsymbol{w}, y)$ is differentiable in $\boldsymbol{w}$. Then $\boldsymbol{z}$ is rationalizable by some production function if and only if,

1. (**Shephard's lemma**) $\nabla_{\boldsymbol{w}} c(\boldsymbol{w}, y) = \boldsymbol{z}(\boldsymbol{w}, y)$.
2. $c(\cdot, y)$ is concave in $\boldsymbol{w}$.

We skipped rationalizability for the EMP and Hicksian demand in Ch. 2. The EMP works on the same logic as the CMP, so the analogous characterization — Shephard's lemma plus concavity in prices — holds for the expenditure function $e(\boldsymbol{p}, u)$ without modification.

Proposition 3.4.3: Properties of Cost Function and Conditional Factor Demand Correspondence

Suppose the production function $f(\cdot)$ is continuous and the production set Y satisfies the free disposal property. Then for $\boldsymbol{w} \gg \mathbf{0}$,

- **Existence of conditional factor demand**: If $Z(y) \neq \varnothing$, then the conditional factor demand correspondence $\boldsymbol{z}(\boldsymbol{w}, y) \neq \varnothing$.
- **Structure of conditional factor demand**: If the production technology is convex (i.e., the upper contour set $\{\boldsymbol{z} \geq \mathbf{0} : f(\boldsymbol{z}) \geq y\}$ is a convex set for any $y \geq 0$), then $\boldsymbol{z}(\boldsymbol{w}, y)$ is a convex set. If the production technology is strictly convex and $Z(y) \neq \varnothing$, then $\boldsymbol{z}(\boldsymbol{w}, y)$ is singleton.
- **Homogeneity**: $c(\boldsymbol{w}, y)$ is homogeneous of degree 1 in $\boldsymbol{w}$, and $\boldsymbol{z}(\boldsymbol{w}, y)$ is homogeneous of degree 0 in $\boldsymbol{w}$. **If the production function $f(\cdot)$ exhibits constant returns to scale, then $c(\boldsymbol{w}, y)$ and $\boldsymbol{z}(\boldsymbol{w}, y)$ are homogeneous of degree 1 in y.**
- **Monotonicity**: $c(\boldsymbol{w}, y)$ is non-decreasing in $\boldsymbol{w}$ and is strictly increasing in y for $y \geq 0$.
- **Binding production level**: For $y > 0$ and $Z(y) \neq \varnothing$, at any minimizer $\boldsymbol{z}^*$, $f(\boldsymbol{z}^*) = y$.
- **Convexity: If $f(\cdot)$ is a concave function, then $c(\boldsymbol{w}, \cdot)$ is a convex function of y.**
- **Shephard's lemma**: If $\boldsymbol{z}(\boldsymbol{w}, y)$ is single-valued, then $c(\boldsymbol{w}, y)$ is differentiable with respect to w_i and $\dfrac{\partial c(\boldsymbol{w}, y)}{\partial w_i} = z_i(\boldsymbol{w}, y)$.

The bolded properties have no counterpart in the EMP. The reason: in consumer theory utility is purely ordinal *— any monotone transformation of u represents the same preferences and gives the same EMP solution. In producer theory the production function f is* cardinal*: it specifies physical output, which has its own units and admits no monotone-transformation freedom. That cardinality is what lets us say, e.g., that the cost function is homogeneous of degree 1 in y under constant returns, or that it is convex in y when f is concave.*

3.4.2 Derivation of Cost Minimization Problem

Definition 3.4.4: Cost Minimization Problem

$$\begin{aligned} &\min_{\boldsymbol{z}} \boldsymbol{w} \cdot \boldsymbol{z} \\ \text{s.t. } & f(\boldsymbol{z}) \geq y \\ & z_i \geq 0, \forall i = 1, 2, \cdots, m \end{aligned}$$

Notice that CMP is almost identical to EMP in consumer theory. The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{z}; \lambda, \boldsymbol{\mu}) = \boldsymbol{w} \cdot \boldsymbol{z} - \lambda(f(\boldsymbol{z}) - y) - \sum_{i=1}^{n} \mu_i z_i.$$

The F.O.C.s are given by:

- w.r.t. z_i: $w_i - \lambda \dfrac{\partial f(\boldsymbol{z})}{\partial z_i} - \mu_i = 0$.
- Inequality constraints: $f(\boldsymbol{z}) \geq y$, $z_i \geq 0$, $\lambda \geq 0$, $\mu_i \geq 0$.
- Complementary slackness: $\lambda (f(\boldsymbol{z}) - y) = 0$, $\mu_i z_i = 0$.

For $y \geq 0$, we have binding production level (i.e., $f(\boldsymbol{z}) = y$), so the F.O.C. can be alternatively framed as

$$\lambda \frac{\partial f(z)}{\partial z_i} \leq w_i, \text{ with equality if } z_i > 0.$$

The economic intuition of $\lambda \dfrac{\partial f(z)}{\partial z_i} \leq w_i$ is that, the LHS corresponds to the marginal benefit, or shadow value of additional one unit of input z_i, and the RHS stands for the marginal cost of such investment. The F.O.C. state that at the optimum, the marginal benefit of inputs cannot exceed their marginal cost. Or more precisely, for those deployed inputs, their marginal benefit just equals marginal cost, while for inputs that are not ever invested, their marginal benefit must be no more than their marginal cost, otherwise the firm still have the room to cut down its cost, indicating the current solution has not reached the optimum.

Remark.

Apply envelope theorem to the Lagrangian of CMP, we have

$$\frac{\partial c(\boldsymbol{w}, y)}{\partial y} = \lambda^*.$$

Specifically, λ^* **measures the firm's marginal cost of producing an additional unit of output** (valued by the firm instead of the market). Hence, another interpretation for the F.O.C. is that, the firm's internal valuation of additional input cannot exceed that of the market. To be specific, the marginal cost of production using factor i, $\frac{w_i}{MP_i}$, should be weakly greater than the marginal cost of producing additional one unit of product, with equality if the factor is employed at the optimum.

With the cost function, we can restate the firm's profit maximization problem in an indirect approach as:

$$\max_{y \geq 0} py - c(\boldsymbol{w}, y)$$

Clearly, the F.O.C. is given by

$$p \leq \frac{\partial c(\boldsymbol{w}, y)}{\partial y}, \text{ with equality if } y > 0.$$

If we relax the assumption of perfect competition and instead assume that the firm is a monopolist in the output market but a price-taker in the input market, then the firm no longer takes the output price as given. We can restate the firm's profit maximization problem as:

$$\max_{y \geq 0} p(y)\, y - c(\boldsymbol{w}, y)$$

The F.O.C. is given by

$$p'(y)\, y + p(y) \geq \frac{\partial c(\boldsymbol{w}, y)}{\partial y}, \text{ with equality if } y > 0.$$

The interpretation is similar to the perfectly competitive case. This indirect approach proves to be useful and attests to the power of cost minimization problem.