Chapter 4

Comparative Statics Analysis

A central question in economics is how an endogenous variable moves with an exogenous parameter. Take the firm's indirect single-product profit maximization:

$$\max_{y \geq 0} \; py - c(\boldsymbol{w}, y).$$

Holding the input price vector $\boldsymbol{w}$ fixed, how does the optimal supply y^* respond to the output price p?

More generally, given an objective $F : X \times \Theta \to \mathbb{R}$ with $X, \Theta \subset \mathbb{R}$, we ask how the maximizer

$$x^*(\theta) = \arg\max_{x \in X} F(x, \theta)$$

moves with the parameter θ. The two competing approaches we will develop are the *classical* (calculus-based, using the implicit function theorem) and the *monotone* or *lattice* approach (order-theoretic, requiring only complementarity). The latter is more robust — it works without differentiability, without concavity, and even when $x^*(\theta)$ is set-valued.

4.1 Univariate Comparative Statics

The classical approach uses the implicit function theorem. It requires two sets of assumptions:

- $F(\cdot, \cdot)$ is twice continuously differentiable.
- The maximizer $x^*(\theta)$ is unique and is characterized by the first-order condition.

Remark.

A set of sufficient conditions for the second assumption is needed:

- The choice set X is convex.
- $F(\cdot, \theta)$ is strictly concave in x.
- The solution is interior.

Under the two sets of assumptions, the first-order condition is:

$$F_x(x(\theta),\theta) = 0.$$

Differentiating on both sides with respect to θ, we have:

$$F_{xx}(x(\theta),\theta)\cdot x'(\theta) + F_{x\theta}(x(\theta),\theta) = 0$$
$$\Longrightarrow x'(\theta) = -\frac{F_{x\theta}(x(\theta),\theta)}{F_{xx}(x(\theta),\theta)}$$

If $F(\cdot,\theta)$ is strictly concave in x, then $F_{xx}(\cdot,\cdot) < 0$ (assuming no reflection point), so $x(\cdot)$ is strictly increasing in θ if $F_{x\theta}(\cdot,\cdot) > 0$.

The classical method's strength is that it delivers a closed-form expression for $x'(\theta)$, which is useful for quantitative work. The flip side is that the same machinery brings real drawbacks:

- *Technical:* the prerequisites (twice differentiability, strict concavity, interior solution) are strong, and the algebra can be tedious.
- *Substantive:* strict concavity in x is a particularly awkward assumption, because concavity is not invariant under monotone transformations — but the comparative-statics *direction* of $x^*(\theta)$ is. Asking for concavity to detect a direction that does not depend on concavity is overkill.

Example.

Taking $\boldsymbol{w}$ as given, the profit maximization problem is given by

$$\max_{y\geq 0} py - c(y)$$

We use the classical approach to determine whether (and when) the firm's supply curve is (weakly) upward sloping.

Notice that the choice set $Y = [0,+\infty)$ is convex, and when $c(\cdot)$ is strictly convex, the objective function is strictly concave. Assuming interiority, the first-order condition is given by:

$$p = c'\left(y\left(p\right)\right)$$

Differentiating on both sides with respect to p, we get

$$y'\left(p\right) = \frac{1}{c''\left(y\left(p\right)\right)}$$

Thus, $c(\cdot)$ being strictly convex is a sufficient condition for the supply curve being upward sloping. However, the requirement of $c(\cdot)$ being strictly convex is not always a reasonable assumption. Moreover, this assumption is not necessary. Recall that we have *Law of Supply*, which states that the firm's supply curve is weakly upward sloping without any other assumption.

The diagnosis: differentiability and concavity of $F(\cdot,\theta)$ are not what comparative statics fundamentally needs. The real driver is $F_{x\theta} \geq 0$ — the *complementarity* between x and

θ. The natural question is whether this cross-partial condition (suitably generalized) is by itself enough for $x^*(\theta)$ to be increasing. The remainder of this section develops a discrete, order-theoretic analogue that does not require F to be differentiable or concave.

Definition 4.1.1: Increasing Differences

Let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$. We say that $F(\cdot,\cdot)$ has *increasing differences* in (x, θ) if for any $x, x' \in X$ and $\theta, \theta' \in \Theta$ such that $x' > x$ and $\theta' > \theta$, we have

$$F(x', \theta') - F(x, \theta') \geq F(x', \theta) - F(x, \theta)$$

If the inequalities are strict for all such $x, x' \in X$ and $\theta, \theta' \in \Theta$, $F(\cdot,\cdot)$ has *strictly increasing differences* in (x, θ).

Intuitively, increasing differences says that the *marginal value* of x is larger at higher θ — x and θ are *complements*. Crucially, the condition is stated in terms of discrete differences, so it makes sense even when F is not differentiable, and it is invariant under monotone transformations of F (unlike concavity).

Proposition 4.1.2: Increasing Differences for Smooth Functions

Suppose $X = [\underline{x}, \bar{x}]$ and $\Theta = \left[\underline{\theta}, \bar{\theta}\right]$, where $X, \Theta \subset \mathbb{R}$.

1. If $F(\cdot,\cdot)$ is continuously differentiable in both x and θ, $F(\cdot,\cdot)$ has increasing differences in (x, θ) if and only if either of the two conditions holds:
   - $F_x(x, \cdot)$ is non-decreasing in θ for all x.
   - $F_\theta(\cdot, \theta)$ is non-decreasing in x for all θ.
2. If $F(\cdot,\cdot)$ is twice continuously differentiable in both x and θ, $F(\cdot,\cdot)$ has increasing differences in (x, θ) if and only if $F_{x\theta}(\cdot,\cdot) \geq 0$ for all (x, θ).

One last piece of plumbing. When $F(\cdot, \theta)$ is not strictly concave, $x^*(\theta)$ is a *set*, not a single number. We need a way to say that one set lies "below" another. Two natural orders on sets serve this purpose:

Definition 4.1.3: Comparison of Sets

For any two sets A and B, we say that:

- $A \leq B$ *in the strong set order* if for any $a \in A$ and $b \in B$, we have $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$.
- $A \leq B$ *pointwise* if for any $a \in A$ and $b \in B$, we have $a \leq b$.

Intuitively, the *strong set order* allows A and B to overlap (the overlap forms a shared region), but outside that overlap every A-element lies below every B-element. The *pointwise* order is stricter: every element of B must lie at or above *every* element of A, regardless of overlap.

Theorem 4.1.4: Univariate Topkis' Theorem

Let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$, and $x^* (\theta) = \arg\max_{x \in X} F (x, \theta)$. Then for any $\theta' > \theta$,

1. If $F (\cdot, \cdot)$ has increasing differences in (x, θ), then $x^* (\theta) \le x^* (\theta')$ in the strong set order.
2. If $F (\cdot, \cdot)$ has strictly increasing differences in (x, θ), then $x^* (\theta) \le x^* (\theta')$ pointwise.

Proof for Theorem

Take any $x \in x^* (\theta)$ and $x' \in x^* (\theta')$. Suppose $x > x'$, then by revealed preference,

$$F (x, \theta) \ge F (x', \theta)$$
$$F (x', \theta') \ge F (x, \theta')$$

By increasing differences,

$$F (x, \theta) - F (x', \theta) \le F (x, \theta') - F (x', \theta')$$

Jointly we have:

$$0 \le F (x, \theta) - F (x', \theta) \le F (x, \theta') - F (x', \theta') \le 0$$
$$\implies \begin{cases} F (x, \theta) = F (x', \theta) \\ F (x, \theta') = F (x', \theta') \end{cases} \implies \begin{cases} x \in x^* (\theta') \\ x' \in x^* (\theta) \end{cases}$$

If $F (\cdot, \cdot)$ has strictly increasing differences in (x, θ), then the combined inequality cannot hold, so it must be that $x^* (\theta) \le x^* (\theta')$ pointwise. In particular, if $F (\cdot, \cdot)$ is twice continuously differentiable in both x and θ and $x^* (\theta)$ is single-valued, then $F_{x\theta} (\cdot, \cdot) > 0$ is sufficient for $x^* (\theta)$ to be (weakly) increasing in θ. ■

Example.

Again consider the firm's profit maximization problem. How would the firm's supply (correspondence) change with the output price p?

The objective function is $F (y, p) = py - c (\boldsymbol{w}, y)$. $Y = [0, +\infty)$ and $P = [0, +\infty)$ are both intervals in $\mathbb{R}$. Moreover, $F_p (y, p) = y$, which is strictly increasing in y, so $F (\cdot, \cdot)$ has strictly increasing differences. Therefore, the firm's supply increases with the output price p pointwise.

Example.

Consider a monopolist that faces a downward sloping demand curve $Q^D (p)$. Now suppose the government levies a unit tax t on the firm. How would the before-tax price p received by the firm change with the unit tax t?

The objective function is $F (p, t) = (p - t) Q^D (p) - c \left(Q^D (p)\right)$. $P = [0, +\infty)$ and $T = [0, +\infty)$ are both intervals in $\mathbb{R}$. Moreover, $F_t (p, t) = -Q^D (p)$, which is strictly increasing in p (since the demand curve is downward sloping), so $F(\cdot,\cdot)$ has strictly increasing differences in (p,t). Consequently, the firm's before-tax price p increases with t pointwise.

Notice that whether $x^*(\theta)$ increases/decreases with the parameter θ is an *ordinal* property, while (strictly) increasing differences is still a *cardinal* property. Indeed, we know from the discussion on consumer theory that $\max_x F(x,\theta)$ and $\max_x G(x,\theta)$ have the same set of maximizers if $G = \varphi \circ F$ for $\varphi(\cdot)$ strictly increasing. Nevertheless, G having increasing differences in (x,θ) does not necessarily mean F having (strictly) increasing differences in (x,θ). In other words, the requirement of (strictly) increasing differences is still too strong for monotone comparative statics. For our purpose, if we can find a positive and monotonic transformation φ such that $G = \varphi \circ F$ has increasing differences or strictly increasing differences in (x,θ), then we know $x^*(\theta)$ increases with θ in the strong set order or pointwise.

Example.

Consider the effects of an increase in the market size on monopoly quantity (and monopoly price). Each consumer in the market has an identical inverse function given by $p^D(q)$. Suppose the number of consumers N is exogenously given, and that the firm's cost function is $c(Q)$, where $Q = Nq$ is the total quantity sold (i.e., the number of consumers times per unit purchase). Discuss how the firm's cost function $c(\cdot)$ would affect the optimal *per-consumer* quantity $q^*(N)$ as the number of consumers N increases.

Solution.

The objective function is

$$F(q,N) = N \cdot p^D(q) \cdot q - c(Nq).$$

Notice that it is hard to check increasing differences of $F(\cdot,\cdot)$. (You may try it yourself and find the process blocked by some terms that need additional information to push forward the computation.) Consider $G(q,N) = \dfrac{F(q,N)}{N}$. Then if $G(\cdot,\cdot)$ is twice continuously differentiable (which indeed can be relaxed), we have

$$\begin{aligned}
G(q,N) &= \frac{F(q,N)}{N} = p^D(q) \cdot q - \frac{c(Nq)}{N} \\
G_N(q,N) &= -\frac{q \cdot c'(Nq) \cdot N - c(Nq)}{N^2} \\
G_{Nq}(q,N) &= -qc''(Nq)
\end{aligned}$$

If $c(\cdot)$ is concave, then $G_{Nq}(q,N) \geq 0$ and $G(q,N)$ has increasing differences in (q,N), so $q^*(\cdot)$ weakly increases with N. If $c(\cdot)$ is convex, then $G_{Nq}(q,N) \leq 0$ and $G(q,N)$ has increasing differences in $(q,-N)$, so $q^*(\cdot)$ weakly decreases with N.

4.2 Multivariate Comparative Statics

Consider a two-variable maximization problem:

$$\left(x_1^*(\theta), x_2^*(\theta)\right) = \arg \max_{(x_1,x_2)\in X\subset\mathbb{R}^2} F(x_1, x_2, \theta).$$

If F merely has increasing differences in (x_1, θ), that alone does not let us conclude that $x_1^*(\theta)$ is weakly increasing — *unless* x_2^* is independent of θ. The reason: when θ moves, it has a direct effect on x_1, but also an *indirect* effect channeled through x_2 (since x_2^* may shift, which in turn changes the optimal x_1). To sign the total effect we need a stronger structure on F that controls both channels at once.

In the univariate case the strong set order used $\min\{a, b\}$ and $\max\{a, b\}$. Extending this to vectors raises two questions:

- How should we define "min" and "max" of two vectors?
- Under that definition, will the result still lie in X?

For the first question, we introduce *meet* and *join* — the componentwise minimum and maximum, respectively, which give the greatest lower bound and the smallest upper bound of two vectors.

Definition 4.2.1: Meet; Join

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

- *meet* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as
$$\boldsymbol{x} \wedge \boldsymbol{y} = \left(\min\{x_1, y_1\}, \min\{x_2, y_2\}, \cdots, \min\{x_n, y_n\}\right).$$
- *join* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as
$$\boldsymbol{x} \vee \boldsymbol{y} = \left(\max\{x_1, y_1\}, \max\{x_2, y_2\}, \cdots, \max\{x_n, y_n\}\right).$$

where $\boldsymbol{x} \wedge \boldsymbol{y}$ is called the greatest lower bound of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{x} \vee \boldsymbol{y}$ is called the smallest upper bound of $\boldsymbol{x}$ and $\boldsymbol{y}$.

For the second question, we restrict attention to choice sets that are *closed under meet and join*. This structure is called a *sublattice*.

Definition 4.2.2: Sublattice

A set $X \subset \mathbb{R}^n$ is a *sublattice* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$, both $\boldsymbol{x} \wedge \boldsymbol{y} \in X$ and $\boldsymbol{x} \vee \boldsymbol{y} \in X$.

$\mathbb{R}^n$ *itself is a* lattice, *so any set in* $\mathbb{R}^n$ *is called a* sublattice.

Example.

- $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subset \mathbb{R}$, for $i = 1, 2, \cdots, n$.
- $X = \left\{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{p} \cdot \boldsymbol{x} \le m\right\}$, where $\boldsymbol{p} \gg \boldsymbol{0}$ is the price vector and $m > 0$ is the income. (NOT a sublattice)

- $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq g(x_1), \text{ for } g(\cdot) \text{ strictly increasing}\right\}$.

The multivariate analogue of increasing differences is *supermodularity* — a single condition that captures complementarity across all pairs of choice variables simultaneously.

Definition 4.2.3: Supermodularity

Let $F : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice. F is *supermodular* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$F(\boldsymbol{x} \wedge \boldsymbol{y}) + F(\boldsymbol{x} \vee \boldsymbol{y}) \geq F(\boldsymbol{x}) + F(\boldsymbol{y}).$$

Remark.

- Note when $X = X_1 \times X_2 \subset \mathbb{R}^2$, supermodularity is equivalent to increasing differences in (x_1, x_2).
- More generally, when $X = X_1 \times X_2 \times \cdots \times X_n \subset \mathbb{R}^n$, **supermodularity is equivalent to increasing differences in (x_i, x_j) for all pairs of $i \neq j$.**

Putting all together, we have the following multivariate Topkis's Theorem.

Proposition 4.2.4: Multivariate Topkis's Theorem

Let $F : X \times \Theta \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice and $\Theta \subset \mathbb{R}$. Consider $\boldsymbol{x}^*(\theta) = \arg\max_{\boldsymbol{x} \in X} F(\boldsymbol{x}, \theta)$. If F is supermodular, then for any $\theta' > \theta$ and $\boldsymbol{x} \in \boldsymbol{x}^*(\theta)$ and $\boldsymbol{x}' \in \boldsymbol{x}^*(\theta')$, we have

$$\boldsymbol{x} \wedge \boldsymbol{x}' \in \boldsymbol{x}^*(\theta),$$
$$\boldsymbol{x} \vee \boldsymbol{x}' \in \boldsymbol{x}^*(\theta').$$

Proof for Proposition.

Since X is a sublattice, $\boldsymbol{x} \wedge \boldsymbol{x}' \in X$ and $\boldsymbol{x} \vee \boldsymbol{x}' \in X$. By revealed preference,

$$\begin{cases} F(\boldsymbol{x}, \theta) \geq F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) \\ F(\boldsymbol{x}', \theta') \geq F(\boldsymbol{x} \vee \boldsymbol{x}', \theta') \end{cases} \Longrightarrow F(\boldsymbol{x}, \theta) + F(\boldsymbol{x}, \theta') \geq F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) + F(\boldsymbol{x} \vee \boldsymbol{x}', \theta')$$

Since F is supermodular,

$$\begin{aligned} F(\boldsymbol{x}, \theta) + F(\boldsymbol{x}', \theta) &\leq F((\boldsymbol{x}, \theta) \wedge (\boldsymbol{x}', \theta')) + F((\boldsymbol{x}, \theta) \vee (\boldsymbol{x}', \theta')) \\ &= F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) + F(\boldsymbol{x} \vee \boldsymbol{x}', \theta') \end{aligned}$$

Jointly, it must be that

$$F(\boldsymbol{x}, \theta) + F(\boldsymbol{x}, \theta') = F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) + F(\boldsymbol{x} \vee \boldsymbol{x}', \theta')$$

Consequently, we have

$$F(\boldsymbol{x} \wedge \boldsymbol{x}', \theta) = F(\boldsymbol{x}, \theta)$$
$$F(\boldsymbol{x} \vee \boldsymbol{x}', \theta') = F(\boldsymbol{x}, \theta')$$

which indicates that $\boldsymbol{x} \wedge \boldsymbol{x}' \in \boldsymbol{x}^*(\theta)$ and $\boldsymbol{x} \vee \boldsymbol{x}' \in \boldsymbol{x}^*(\theta')$. ∎

Example.

Suppose a firm uses two inputs (L and K) to produce a single output y, and the production function is given by

$$y = f(L, K)$$

where $f(\cdot, \cdot)$ is twice continuously differentiable.

1. Discuss how the optimal demand for capital $K^*(w, r, p)$ would be affected by an increase in the rental rate of the capital r.
2. Discuss how the optimal demand for labor $L^*(w, r, p)$ would be affected by an increase in the rental rate of the capital r.

Solution.

$$\pi(L, K, r) = pf(L, K) - wL - rK.$$

Immediately, $\pi_{Lr} = 0$ and $\pi_{Kr} = -1$, π is supermodular in $(-K, r)$, so $K^*(w, r, p)$ weakly decreases with r. Next consider the indirect path of r to L through K.

- If $f_{LK} > 0$, π is supermodular in $(-L, -K, r)$, so $L^*(w, r, p)$ weakly decreases with r.
- If $f_{LK} < 0$, π is supermodular in $(L, -K, r)$, so $L^*(w, r, p)$ weakly increases with r.

Example.

Consider the other form of market power, a market with a single buyer and competitive supply. Suppose the inverse supply curse $p^S(q) \geq 0$ is strictly increasing. Let $v(q)$ be the buyer's value for quantity q. Consequently, if there are q units of transaction in the market,

- The producer surplus is

$$S(q) = \int_0^q \left[p^S(q) - p^S(t)\right] \, \mathrm{d}t.$$

- The buyer's gain is

$$B(q) = v(q) - p^S(q)q.$$

1. Formulate both the buyer's and the social planner's optimization problems (no need to solve either of them).
2. How does the single buyer's optimal quantity q^B compare with that of the social planner q^*? Show your argument and explain intuitively.

Solution.

1. The buyer's optimization problem is

$$\max_{q \geq 0} v(q) - p^S(q)q.$$

The social planner's optimization problem is

$$\max_{q\geq 0} v(q) - \int_0^q p^S(t)\ \mathrm{d}t.$$

2. Consider the following optimization problem:

$$\max_{q\geq 0} w(q,\lambda) = B(q) + \lambda S(q)$$

This becomes the buyer's optimization problem when $\lambda = 0$, and the social planner's when $\lambda = 1$. If we could show how q monotonically change with λ, then we are done with this question. This is the univariate comparative statics problem, so we check the increasing differences of $w(q,\lambda)$:

$$\frac{\partial w(q,\lambda)}{\partial \lambda} = S(q)$$

Take any $q' \geq q \geq 0$. We have

$$\begin{aligned} S(q') - S(q) &= \int_0^{q'} \left[p^S(q') - p^S(t)\right]\ \mathrm{d}t - \int_0^{q} \left[p^S(q) - p^S(t)\right]\ \mathrm{d}t \\ &= \int_q^{q'} \left[p^S(q') - p^S(q)\right]\ \mathrm{d}t + \int_0^{q} \left[p^S(q') - p^S(q)\right]\ \mathrm{d}t \\ &= \int_q^{q'} \left[p^S(q') - p^S(q)\right]\ \mathrm{d}t + \left[p^S(q') - p^S(q)\right] q \\ &\geq 0 \end{aligned}$$

where the last inequality holds because $p^S(\cdot)$ is upward sloping.

Thus, $w(q,\lambda)$ has increasing differences in (q,λ). From Topkis's Theorem, $q^*(\lambda) < q^*(\lambda')$.

Similar to increasing differences, supermodularity is a *cardinal* property, which is again too strong. Indeed, the weaker requirement, *quasi-supermodularity* serves our purpose.

Definition 4.2.5: Quasi-Supermodularity

Let $F : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice. F is *quasi-supermodular* if for any $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$F(\boldsymbol{x}) \geq F(\boldsymbol{x} \wedge \boldsymbol{y}) \Longrightarrow F(\boldsymbol{x} \vee \boldsymbol{y}) \geq F(\boldsymbol{y}),$$
$$F(\boldsymbol{x}) > F(\boldsymbol{x} \wedge \boldsymbol{y}) \Longrightarrow F(\boldsymbol{x} \vee \boldsymbol{y}) > F(\boldsymbol{y}).$$

Under such extension, the analysis can be further extended beyond the case of $X \subset \mathbb{R}^n$ and X forming a lattice structure.