# Chapter 6

# General Equilibrium

The unifying idea across all market analyses is *market clearing*. What distinguishes the two standard approaches is how many markets they ask to clear at once. *Partial equilibrium* fixes prices in all other markets and analyzes a single market in isolation. *General equilibrium* requires every market in the economy to clear simultaneously, taking into account the cross-market price linkages.

Partial equilibrium remains a useful shortcut despite its narrower scope. It is a defensible approximation whenever:

- Prices in all other markets are essentially unaffected by what happens in the market of interest (e.g., the market is small relative to the rest of the economy).
- There are no significant wealth effects feeding back from the market of interest into demand elsewhere.

The following example shows what can go wrong when these conditions fail.

**Example.**

There are $n$ towns in total, with $n$ being a large number. Each town has an identical price-taking firm that produces a single consumption good with the production function $y = f(l)$, where $f'(l) > 0$ and $f''(l) < 0$. The single consumption good is traded in the national market. Suppose there are $L$ units of inelastic labor supply in total. Workers can move freely across towns to seek the highest possible wage, which are regarded as complete information. Normalize $p_c = 1$ and let $w_i$ denote the equilibrium wage in town $i$. Now suppose that town 1 levies a **small** tax $t > 0$ on firm 1 for each unit of labor hired. Which group of individual(s) will bear the tax burden?

**Solution.**

- Equilibrium without tax
  - Whether we adopt the partial or general equilibrium approach, the competitive equilibrium without the labor tax is identical.
  - Since workers can move freely and $f'(l) > 0$ and $f''(l) < 0$, we must have

$$\begin{cases} l^* = \frac{1}{n} \cdot L \\ w_1 = w_2 = \cdots = w_n = f'(l^*) = f'\left(\frac{l}{n}\right) \end{cases}$$

- Partial equilibrium in town 1 with tax
  - Assuming the wage rate in other markets are unaffected, we must have $w_1(t) = w_0 + t$.
  - At the new equilibrium, $f'(l_1(t)) = w_1(t) = w_0 + t$.
  - Hence we can see that firm 1 bears all the tax burden.
- General equilibrium with town 1 levied with tax
  - Let $l_1(t)$ denote the equilibrium amount of labor in town 1 with a $t$ unit tax, and $l_{-1}(t)$ denote the equilibrium amount of labor in any other town when town 1 imposes a $t$ unit tax and $w(t)$ the equilibrium wage rate received by the worker. When all labor markets clear, we should have:

$$\begin{cases} l_1(t) + (n-1)\, l_{-1}(t) = L \\ f'(l_1(t)) = w_1(t) = w(t) + t \\ f'(l_{-1}(t)) = w(t) \end{cases}$$
$$\Longrightarrow \begin{cases} f'(L - (n-1)\, l_{-1}(t)) = w(t) + t \\ f'(l_{-1}(t)) = w(t) \end{cases}$$
$$\Longrightarrow \begin{cases} -(n-1)\, l'_{-1}(t) \cdot f''(L - (n-1)\, l_{-1}(t)) = w'(t) + 1 \\ l'_{-1}(t)\, f''(l_{-1}(t)) = w'(t) \end{cases}$$

  - Set $t \to 0$, so we have $l_{-1}(0) = \frac{1}{n} \cdot L$. From the two equations above we obtain $w'(0) = -\frac{1}{n}$.
  - Let $\Pi(w(t))$ denote the total profit of the firms when the wage rate is $w(t)$ and $\pi(w)$ the profit of a firm paying wage $w$. Naturally, $\Pi(w(t)) = \pi(w(t) + t) + (n-1)\,\pi(w(t))$, and

$$\frac{\partial \Pi(w(t))}{\partial t} = (w'(t) + 1) \cdot \pi'(w(t) + t) + (n-1)\, w'(t) \cdot \pi'(w(t))$$
$$\Longrightarrow \left.\frac{\partial \Pi(w(t))}{\partial t}\right|_{t \to 0} = (w'(0) + 1) \cdot \pi'(w(0)) + (n-1)\, w'(0) \cdot \pi'(w(0)) = 0$$

  - From quantitative analysis we can see that, the firms as a whole do not bear the tax burden and the workers bear all the tax burden.
  - Intuitively, it must be the workers that bear all the tax burden. Even though the labor supply for any firm is perfectly elastic, the **total** labor supply is **perfectly inelastic**. Small as the impact on the wages in other towns, it is not negligible. Indeed, the labor supply for any given firm is perfectly elastic, so we cannot ignore the general equilibrium effect.

## 6.1 Pure Exchange Economy

A full market analysis typically involves three activities: consumption, production, and trade. The cleanest setting to develop general-equilibrium intuition strips out production: in a *pure exchange economy*, each agent starts with an endowment of goods, and the only economic activity is mutually beneficial trade.

**Example.**

Consider a perfectly competitive economy with two agents ($i = A, B$) and two goods ($j = 1, 2$). Suppose the agents' preference relations and initial endowments are given by

$$u^A(x_1, x_2) = x_1^2 x_2 \text{ with } e^A = (1, 2)$$
$$u^B(x_1, x_2) = x_1 x_2^2 \text{ with } e^B = (2, 1)$$

Can we come up with a price vector $(p_1, p_2)$ that clear both markets?

**Solution.**

The key idea is that, each agent has their own income, which is endogenously given by the equilibrium price vector.

Suppose there is an equilibrium price vector $(p_1, p_2)$, then agent $A$'s "income" is $m^A = p_1 + 2p_2$ and agent $B$'s "income" is $m^B = 2p_1 + p_2$. From this we can pin down the optimal individual demands

$$\boldsymbol{x}^A = \left(\frac{2(p_1 + 2p_2)}{3p_1}, \frac{1(p_1 + 2p_2)}{3p_2}\right)$$
$$\boldsymbol{x}^B = \left(\frac{1(2p_1 + p_2)}{3p_1}, \frac{2(2p_1 + p_2)}{3p_2}\right)$$

In equilibrium, market demand equals market endowment for either good, respectively:

$$\frac{2(p_1 + 2p_2)}{3p_1} + \frac{1(2p_1 + p_2)}{3p_1} = 3$$
$$\frac{1(p_1 + 2p_2)}{3p_2} + \frac{2(2p_1 + p_2)}{3p_2} = 3$$

From either equation, we have $\frac{p_1^*}{p_2^*} = 1$. (Otherwise if there comes with a contradiction, the equilibrium cannot exist.)

From this example, we obtain four observations in this two-agent, two-good pure exchange economy:

1. There is an equilibrium price vector that clears both markets.
2. Only the relative price matters in equilibrium.
3. When one market clears, the other market clears simultaneously (and miraculously).
4. The equilibrium allocation is Pareto efficient.

### 6.1.1 Setups

**Model Preliminaries**

- $I = \{1, 2, \cdots, n\}$ agents and $J = \{1, 2, \cdots, m\}$ goods.
- $\succsim^i$: preference relation of agent $i$.
- $\mathbf{e}^i = \left\{e_1^i, e_2^i, \cdots, e_m^i\right\}$ initial endowment of agent $i$ (property rights are well-defined and no co-ownership).
- Denote the economy $\mathcal{E} = \left\{\succsim^i, \mathbf{e}^i\right\}_{I \in \mathcal{I}}$.

**Assumption 6.1.1: Market Structure**

1. Perfect and complete information.
2. Perfectly competitive markets.
   - Agents are price-takers.
   - Prices are linear.
3. Goods are perfectly divisible.

**Assumption 6.1.2: Agents' Preferences and Endowments**

For any agent $i \in \mathcal{I}$:

1. The preference relation $\succsim^i$ is rational (complete and transitive) and continuous.
2. The preference relation $\succsim^i$ is monotonic.
3. The preference relation $\succsim^i$ is (weakly) convex.
4. $e_j^i > 0$, for all $j \in \mathcal{J}$.

Notice that by the first assumption, the utility representation of $\succsim^i$, say $u^i(\cdot)$, is guaranteed for any $i \in I$, so we can alternatively denote the economy as $\mathcal{E} = \left(u^i(\cdot), \boldsymbol{e}^i\right)_{i \in \mathcal{I}}$.

### 6.1.2 Walrasian Equilibrium

**Definition 6.1.3: Walrasian Equilibrium**

A *Walrasian Equilibrium* for a perfectly competitive pure exchange economy $\mathcal{E} = \left(u^i\left(\cdot\right), \mathbf{e}^i\right)_{i\in\mathcal{I}}$ is a price vector $\boldsymbol{p} \geq \mathbf{0}$ and an allocation $\left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}$ such that:

1. Utility maximization: Agent $i \in \mathcal{I}$ solves:

$$\max_{\boldsymbol{x}^i \geq \mathbf{0}} u^i\left(\boldsymbol{x}^i\right) \text{ s.t. } \boldsymbol{p}\cdot\boldsymbol{x}^i \leq \boldsymbol{p}\cdot\mathbf{e}^i$$

2. Market clearing for all goods:

$$\sum_{i\in\mathcal{I}} \boldsymbol{x}^i\left(\boldsymbol{p}\right) = \sum_{i\in\mathcal{I}} \mathbf{e}^i$$

Under Walrasian equilibrium, all markets clear at the same time. A Walrasian equilibrium specifies both *equilibrium price vector* and *equilibrium allocation.*

For simplicity, suppose for any $i \in \mathcal{I}$, $\boldsymbol{x}^i\left(\boldsymbol{p}\right)$ is always unique. This is guaranteed if we assume each $\succsim_i$ is strictly convex and $\boldsymbol{p} \gg \mathbf{0}$.

**Definition 6.1.4: Excess Demand**

The (aggregate) *excess demand* for good $j$ is given by

$$z_j\left(\boldsymbol{p}\right) = \sum_{i\in\mathcal{I}} x_j^i\left(\boldsymbol{p}\right) - \sum_{i\in\mathcal{I}} e_j^i$$

With this formulation, a Walrasian equilibrium is achieved when each agent maximize their utility subject to budget constraint prescribed by their endowment, and

$$z_j\left(\boldsymbol{p}\right) = 0, \forall j \in \mathcal{J}$$

**Proposition 6.1.5: Properties of Excess Demand**

Let $\mathcal{E}$ be a perfectly competitive pure exchange economy with each agent's preference relation being rational, continuous and monotonic, then the aggregate excess demand function $\boldsymbol{z}\left(\boldsymbol{p}\right)$ satisfies:

1. **Homogeneity of degree** 0: $z_j\left(\boldsymbol{p}\right)$ is homogeneous of degree 0 in $\boldsymbol{p}$, for any $j \in \mathcal{J}$.

2. **Walras' law**: $\boldsymbol{p}\cdot\boldsymbol{z}\left(\boldsymbol{p}\right) = 0$.

***Proof for Proposition.***

- Homogeneous of degree 0
  - $\boldsymbol{x}^i\left(\boldsymbol{p}\right)$ solves $\max_{\boldsymbol{x}^i\geq\mathbf{0}} u^i\left(\boldsymbol{x}^i\right)$ s.t. $\boldsymbol{p}\cdot\boldsymbol{x}^i \leq \boldsymbol{p}\cdot\boldsymbol{e}^i$.
  - For any $t > 0$, the solution must be the same as: $\max_{\boldsymbol{x}^i\geq\mathbf{0}} u^i\left(\boldsymbol{x}^i\right)$ s.t. $\left(t\boldsymbol{p}\right)\cdot\boldsymbol{x}^i \leq$

$(t\boldsymbol{p}) \cdot \boldsymbol{e}^i$. Hence, $\boldsymbol{x}^i(\boldsymbol{p}) = \boldsymbol{x}^i(t\boldsymbol{p})$.

– For any $t > 0$, $z_j(t\boldsymbol{p}) = \sum_{i=1}^n x_j^i(t\boldsymbol{p}) - \sum_{i=1}^n e_j^i = \sum_{i=1}^n x_j^i(\boldsymbol{p}) - \sum_{i=1}^n e_j^i = z_j(\boldsymbol{p})$.

- Walras' law
  - Budget constraint must be binding for any agent $i \in \mathcal{I}$, $\boldsymbol{p} \cdot \boldsymbol{x}^i(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{e}^i$. (This must hold because of assumption of monotonicity of preference relation, which is stronger than locally non-satiation).
  - It follows that

$$\begin{aligned}\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) &= \boldsymbol{p} \cdot \left(\sum_{i=1}^n \boldsymbol{x}^i(\boldsymbol{p}) - \sum_{i=1}^n \boldsymbol{e}^i\right)\\ &= \sum_{i=1}^n \left(\boldsymbol{p} \cdot \boldsymbol{x}^i(\boldsymbol{p}) - \boldsymbol{p} \cdot \boldsymbol{e}^i\right)\\ &= 0\end{aligned}$$

■

The following proposition, the existence of Walrasian equilibrium, is characterized as the cornerstone of modern economics.

**Proposition 6.1.6: Existence of Walrasian Equilibrium**

Let $\mathcal{E} = \left(u^i(\cdot), \boldsymbol{e}^i\right)_{i\in\mathcal{I}}$ be a perfectly competitive pure exchange economy that satisfies all the four assumptions on agents' preferences and endowments. Then a Walrasian equilibrium exists.

***Proof for Proposition.***

We give proof for the case of $m = 2$ and $\succsim^i$ being strictly convex for any $i \in \mathcal{I}$. Given rationality, continuity and strict convexity of $\succsim^i$, $\boldsymbol{x}^i(p_1, p_2)$ is always unique for any $\boldsymbol{p} \gg \boldsymbol{0}$. When $m = 2$, Walras' law states that

$$p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) = 0.$$

Then a Walrasian equilibrium is given by $z_1(p_1, p_2) = 0$ or $z_2(p_1, p_2) = 0$. By monotonicity and strictly convexity of preference relation, the only possibility is $p_1, p_2 > 0$ in equilibrium, so we can normalize $p_1 = 1$ or $p_2 = 1$.

By theorem of the maximum, given rationality, continuity, monotonicity and strict convexity of preference relation, $x_i^1(p_1, 1)$ is continuous for any $i \in \mathcal{I}$. Hence, $z_1(p_1, 1)$ is continuous on $(0, +\infty)$. Given monotonicity and strict convexity, $u^i(\cdot)$ is strongly monotone. Therefore, $\lim_{p_1 \to 0+} x_1^i(p_1, 1) = +\infty$ and $\lim_{p_1 \to 0+} z_1(p_1, 1) = +\infty$. Symmetrically, we have $\lim_{p_2 \to 0+} x_2^i(1, p_2) = +\infty$ and $\lim_{p_2 \to 0+} z_2(1, p_2) = +\infty$. By continuity, $\exists 0 < \underline{p} < \bar{p}$ such that $z_1(\underline{p}, 1) > 0 > z_1(\bar{p}, 1)$. By the intermediate value theorem, $\exists p_1^* \in (\underline{p}, \bar{p})$ such that $z_1(p_1^*, 1) = 0$. ■

## 6.2 Allocation

Recall the two cornerstone results from intermediate micro:

- *First Welfare Theorem.* Any Walrasian equilibrium allocation is Pareto efficient.
- *Second Welfare Theorem.* Any Pareto efficient allocation can be supported as a Walrasian equilibrium after a suitable redistribution of initial endowments.

We now state these rigorously and strengthen the first. Begin with the formal definition of a feasible allocation.

### Definition 6.2.1: Feasible Allocation

For a pure exchange economy $\mathcal{E}$, an allocation $\left(\boldsymbol{x}^i\right)_{i=1}^n$ with $\boldsymbol{x}^i \geq \mathbf{0}$ for all $i$ is *feasible* if for any good $j$,

$$\sum_{i=1}^{n} x_j^i \leq \sum_{i=1}^{n} e_j^i.$$

In other words, total consumption of every good cannot exceed the total endowment of that good.

### Definition 6.2.2: Pareto Efficient Allocation

Given an economy $\mathcal{E}$, a feasible allocation $\left(\boldsymbol{x}^i\right)_{i=1}^n$ is (strongly) *Pareto efficient* if there is no other feasible allocation $\left(\boldsymbol{w}^i\right)_{i=1}^n$ such that $\boldsymbol{w}^i \succsim^i \boldsymbol{x}^i$ for all $i \in \mathcal{I}$, with $\boldsymbol{w}^k \succ^k \boldsymbol{x}^k$ for some $k \in \mathcal{I}$.

A Pareto efficient allocation is one from which no agent can be made strictly better off without making at least one other agent strictly worse off.

### Definition 6.2.3: Core Allocation

A feasible allocation $\left(\boldsymbol{x}^i\right)_{i=1}^n$ is *in the core* of $\mathcal{E}$ if there is no group $\mathcal{I}_0 \subseteq \mathcal{I}$ and an alternative allocation $\left(\boldsymbol{w}^i\right)_{i=1}^n$ such that:

1. For any $j \in \mathcal{J}$, $\sum_{i\in\mathcal{I}_0} w_j^i \leq \sum_{i\in\mathcal{I}_0} e_j^i$.
2. $\boldsymbol{w}^i \succsim^i \boldsymbol{x}^i$ for all $i \in \mathcal{I}_0$, with $\boldsymbol{w}^k \succ^k \boldsymbol{x}^k$ for some $k \in \mathcal{I}_0$.

An allocation is in the core if *no* group of agents — large or small — can find a redistribution of their own endowments that makes every member of the group at least as well off and some member strictly better off. Core allocations are Pareto efficient (taking the "grand coalition" as the blocking group recovers the Pareto condition), but Pareto efficiency does not imply the core: an allocation can be Pareto efficient overall yet have a strictly sub-coalition that could improve on it among themselves.

**Theorem 6.2.4: Strengthened First Theorem of Welfare Economics**

Let $\mathcal{E}$ be a perfectly competitive pure exchange economy in which each agent's preference relation is rational, continuous and concave. Denote $S_P$ as the set of Pareto efficient allocations, $S_C$ the set of core allocations, and $S_W$ the set of Walrasian equilibrium allocations. Then

$$S_W \subseteq S_C \subseteq S_P$$

***Proof for Theorem***

By definition, $S_C \subseteq S_P$, so it suffices to show $S_W \subseteq S_C$. Suppose not, then there exists a Walrasian equilibrium $\left(\boldsymbol{p}; \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}\right)$, a group $\mathcal{I}_0 \subseteq \mathcal{I}$ and an alternative allocation $\left(\boldsymbol{w}^i\right)_{i\in\mathcal{I}}$ such that:

$$\begin{cases} \boldsymbol{w}^i \succsim^i \boldsymbol{x}^i, \text{ for all } i \in \mathcal{I}_0 \\ \boldsymbol{w}^k \succ^k \boldsymbol{x}^k, \text{ for some } k \in \mathcal{K}_0 \subseteq \mathcal{I}_0 \end{cases}$$

By direct revealed preference,

$$\begin{cases} \boldsymbol{p}\cdot\boldsymbol{w}^k > \boldsymbol{p}\cdot\boldsymbol{x}^k, \text{ for all } k \in \mathcal{K}_0 \\ \boldsymbol{p}\cdot\boldsymbol{w}^i \geq \boldsymbol{p}\cdot\boldsymbol{x}^i, \text{ for all } i \in \mathcal{I}_0\backslash\mathcal{K}_0 \end{cases}$$

Summing over $i \in \mathcal{I}_0$, we have

$$\begin{aligned} &\boldsymbol{p}\cdot\sum_{i\in\mathcal{I}_0}\boldsymbol{w}^i > \boldsymbol{p}\cdot\sum_{i\in\mathcal{I}_0}\boldsymbol{x}^i = \boldsymbol{p}\cdot\sum_{i\in\mathcal{I}_0}\boldsymbol{e}^i \\ \Longrightarrow &\sum_{i\in\mathcal{I}_0}\boldsymbol{w}^i \nleq \sum_{i\in\mathcal{I}_0}\boldsymbol{e}^i \end{aligned}$$

■

**Remark.**

- The standard First Welfare Theorem gives $S_W \subseteq S_P$; the strengthened version shows the tighter $S_W \subseteq S_C$.
- $S_W \subseteq S_C$ says that any Walrasian equilibrium is not only efficient but also *coalition-proof*: no group can do better for itself by trading internally. This is a stronger fairness justification for the price mechanism than mere Pareto efficiency.
- The theorem demands very little of preferences (rationality, continuity, concavity). What it does need is the institutional scaffolding: perfect competition, complete information, no externalities, complete markets. Drop any of these and the conclusion can fail.

**Theorem 6.2.5: Second Theorem of Welfare Economics**

Let $\mathcal{E}$ be a perfectly competitive pure exchange economy in which each agent's preference relation is rational, continuous and concave. If $\boldsymbol{x}^i \gg \mathbf{0}$ is Pareto efficient, for all $i$, then there exists an initial endowment $\left(\boldsymbol{e}^i\right)_{i\in\mathcal{I}}$ and a price vector $\boldsymbol{p} \geq \mathbf{0}$ such that $\left(\boldsymbol{p}; \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium given these endowments.

The Second Welfare Theorem is, in a sense, less operational than the First. To support a given Pareto-efficient allocation, the planner needs both the political authority to redistribute initial endowments freely *and* full information about every agent's preferences — neither of which is generally available in practice.

## 6.3 General Equilibrium with Production

### 6.3.1 Setups

Recall that in producer theory, we used production sets and ownership shares to describe the firms and their production technologies.

Suppose there are $K$ firms ($\mathcal{K} = \{1, 2, \cdots, K\}$) in the economy, each firm $k \in \mathcal{K}$ with its production set $Y^k$. Let $\alpha^{ki} \geq 0$ be agent $i$'s ownership share of firm $k$. The production economy can then be described as

$$\mathcal{E} = \left(\left(u^i, \boldsymbol{e}^i, \left(\alpha^{ki}\right)_{k\in\mathcal{K}}\right)_{i\in\mathcal{I}}, \left(Y^k\right)_{k\in\mathcal{K}}\right).$$

**Definition 6.3.1: Walrasian Equilibrium with Production**

A **Walrasian Equilibrium** for a perfectly competitive production economy $\mathcal{E} = \left(\left(u^i, \boldsymbol{e}^i, \left(\alpha^{ki}\right)_{k\in\mathcal{K}}\right)_{i\in\mathcal{I}}, \left(Y^k\right)_{k\in\mathcal{K}}\right)$ is a vector $\left(\boldsymbol{p}, \left(\boldsymbol{x}^i\right)_{i\in\mathcal{I}}, \left(\boldsymbol{y}^k\right)_{k\in\mathcal{K}}\right)$ such that:

1. **Profit maximization**: Firm $k$ solves

$$\max_{y^k \in Y^k} \boldsymbol{p} \cdot \boldsymbol{y}^k$$

2. **Utility maximization**: Agent $i$ solves

$$\max_{\boldsymbol{x}^i \geq \mathbf{0}} u^i\left(\boldsymbol{x}^i\right) \text{ s.t. } \boldsymbol{p} \cdot \boldsymbol{x}^i \leq \boldsymbol{p} \cdot \boldsymbol{e}^i + \sum_{k\in\mathcal{K}} \alpha^{ik} \boldsymbol{p} \cdot \boldsymbol{y}^k$$

3. **Market clearing for all goods**:

$$\sum_{i\in\mathcal{I}} \boldsymbol{x}^i\left(\boldsymbol{p}\right) = \sum_{i\in\mathcal{I}} \boldsymbol{e}^i + \sum_{k\in\mathcal{K}} \boldsymbol{y}^k\left(\boldsymbol{p}\right)$$

That is, a Walrasian equilibrium specifies prices, consumption bundles, and production plans such that every agent maximizes utility (taking firm profits as given dividend income), every firm maximizes profit, and all markets clear simultaneously.

The conceptual structure is identical to the pure-exchange case; production simply adds the firms' optimization problems. For the existence theorem to go through, however, we need three regularity conditions on production technologies.

**Assumption 6.3.2: Production Technology**

For each firm $k \in \mathcal{K}$,

1. $Y^k \neq \varnothing$ is **closed and convex**.
2. **Shutdown and free disposal**: that is, $\mathbf{0} \in Y^k$, and $\boldsymbol{y} \in Y^k$ implies $\boldsymbol{y}' \in Y^k$, for all $\boldsymbol{y}' \leq \boldsymbol{y}$.
3. **Irreversibility**: Let $Y = \bigcup_{k \in \mathcal{K}} Y^k$, then $Y \cap (-Y) = \{\mathbf{0}\}$.

**Remark.**

- $Y^k \neq \varnothing$ is innocuous, otherwise since firm $k$ is allowed to produce nothing, we can just discard $Y^k$ if $Y^k = \varnothing$.
- Irreversibility says that the production cannot be completely reversed.

### 6.3.2 Existence of Walrasian Equilibrium with Production

**Theorem 6.3.3: Existence of Walrasian Equilibrium with Production**

Let $\mathcal{E} = \left(\left(u^i, \boldsymbol{e}^i, (\alpha^{ki})_{k \in \mathcal{K}}\right)_{i \in \mathcal{I}}, (Y^k)_{k \in \mathcal{K}}\right)$ be a perfectly competitive production economy satisfying all the assumptions on preference relation and endowments, and those on production technology. Then a Walrasian equilibrium $\left(\boldsymbol{p}, (\boldsymbol{x}^i)_{i \in \mathcal{I}}, (\boldsymbol{y}^k)_{k \in \mathcal{K}}\right)$ exists.

**Remark.**

- This proposition of existence is the extension of the existence result on pure exchange economies.
- The first and second theorem of welfare economics also generalize to economics with production.

Recall that when a production technology exhibits constant returns to scale, the firm's profit must be either 0 or $+\infty$. For market clearing to make sense, it must be that each firm is making 0 profit.

For simplicity, consider an economy with two goods. The production technology can linearly transform $b$ units of good 1 into (at most) $c$ units of good 2. The production set can then be depicted as

$$Y = \left\{ (y_1, y_2) \,|y_1 \leq 0, y_2 \geq 0, y_2 \leq -\frac{c}{b} y_1 \right\}.$$

Alternatively, we can present the production technology with the vector $a = \left(-1, \frac{c}{b}\right)$.

If a linear activity production $a = (a_1, a_2)$ is ever used in a Walrasian equilibrium (i.e., $\lambda^* > 0$), then it must be that (because the firm is making zero profit)

$$(p_1^*, p_2^*) \cdot (a_1, a_2) = 0.$$

**Example.**

Consider a perfectly competitive economy $\mathcal{E}$ with two goods (1 and 2) and two agents ($A$ and $B$). The agents' utility functions and initial endowments are as follows:

$$u^A (x_1, x_2) = x_1 x_2, \ \boldsymbol{e}^A = (1, 0)$$
$$u^B (x_1, x_2) = x_1 x_2^2, \ \boldsymbol{e}^B = (0, 1)$$

First suppose the economy is pure exchange with no production.

1. Derive the set of Pareto efficient allocations.
2. Derive the set of core allocations.
3. Derive a Walrasian equilibrium $\left((p_1, p_2); \left(x_1^A, x_2^A, x_1^B, x_2^B\right)\right)$.

Next we introduce production: Suppose there is a perfectly competitive firm which can transform one unit of good 1 into (at most) one unit of good 2, i.e., $\mathbf{a} = (-1, 1)$.

4. Derive a Walrasian equilibrium with production $\left((p_1, p_2,); \left(x_1^A, x_2^A, x_2^B, x_2^B\right), \lambda\right)$.

**Solution.**

1. Since both agents' preference relations are monotonic, any Pareto efficient allocation must satisfy

$$\boldsymbol{x}^A + \boldsymbol{x}^B = \boldsymbol{e}^A + \boldsymbol{e}^B$$

Clearly, $((1, 1), (0, 0))$ and $((0, 0), (1, 1))$ are Pareto efficient.

*$((1, 0), (0, 1))$ or $((0, 1), (1, 0))$ are not necessarily Pareto efficient, though we cannot make any agent better off material-wise without making the other worse off material-wise. However, agents' well-being is measured in terms of magnitude of utilities.*

Apart from corner solutions, by the idea of no gain from trade in equilibrium, we must have

$$|MRS^A| = |MRS^B|$$

Therefore, the set of efficient allocations is given by

$$S_P = \left\{ \left(x_1^A, x_2^A, x_1^B, x_2^B\right) \geq \mathbf{0} : x_1^A x_2^B = 2 x_2^A x_1^B, x_1^A + x_1^B = x_2^A + x_2^B = 1 \right\}.$$

It has been checked that $((1,1),(0,0))$ and $((0,0),(1,1))$ are included in the set $S_P$.

2. When there are only two agents, the only subgroups other than the grand coalition are $\{1\}$ and $\{2\}$. In other words, on the basis of Pareto efficiency, the additional requirement for an allocation to be in the core is that no agent is made worse in any allocation compared to their own endowment, which is also termed *individual rationality*. Since the initial endowment is the worst possible allocation for each agent, the set of core allocations $S_C = S_P$.

3. Suppose there is an equilibrium price vector $(p_1, p_2)$. Then agent A's "income" $m^A = p_1$ and agent B's "income" $m^B = p_2$. The optimal individual demands are

$$\begin{aligned}\left(x_1^A, x_2^A\right) &= \left(\frac{1}{2}, \frac{p_1}{2p_2}\right)\\ \left(x_1^B, x_2^B\right) &= \left(\frac{p_2}{3p_1}, \frac{2}{3}\right)\end{aligned}$$

In equilibrium, market demand equals total endowments for each good.

$$\begin{cases}\frac{1}{2} + \frac{p_2}{3p_1} = 1\\ \frac{p_1}{2p_2} + \frac{2}{3} = 1\end{cases} \implies \frac{p_1^*}{p_2^*} = \frac{2}{3}$$

Hence, a Walrasian equilibrium is $\left(\boldsymbol{p} = (2,3), \boldsymbol{x}^A = \left(\frac{1}{2}, \frac{1}{3}\right), \boldsymbol{x}^B = \left(\frac{1}{2}, \frac{2}{3}\right)\right)$.

*Here "**a** Walrasian equilibrium" is emphasized because only relative prices matter in Walrasian equilibrium; theoretically there could be infinitely-many equilibria in terms of absolute prices.*

4. Suppose the production technology is used in equilibrium, then we must have $(p_1, p_2) \cdot \mathbf{a} = \mathbf{0}$, that is, $p_1 = p_2 \iff \frac{p_1}{p_2} = 1$.

In equilibrium, market demand equals market supply for each good.

$$\begin{cases}\frac{1}{2} + \frac{1}{3} = 1 - \lambda\\ \frac{1}{2} + \frac{2}{3} = 1 + \lambda\end{cases} \implies \lambda^* = \frac{1}{6}$$

A Walrasian equilibrium is given by $\left(\boldsymbol{p}^* = (1,1), \boldsymbol{x}^A = \left(\frac{1}{2}, \frac{1}{2}\right), \boldsymbol{x}^B = \left(\frac{1}{3}, \frac{2}{3}\right), \lambda^* = \frac{1}{6}\right)$.