Part I

Foundations

Part II

Bargaining

Chapter 3

Two-Person Bargaining

3.1 Nash Bargaining Solution

Definition 3.1: Bargaining Problem

A **(two-person) bargaining problem** is a pair (U, d), where $U \subseteq \mathbb{R}^2$ is the *feasible utility set*—a compact, convex set of expected utility profiles attainable by the two parties—and $d \in U$ is the *disagreement point*, the outcome that arises if no agreement is reached. We assume non-degeneracy: there exists some $u \in U$ with $u \gg d$ (i.e., $u_i > d_i$ for both i).

Definition 3.2: Bargaining Solution

A **bargaining solution** is a function F that assigns to every bargaining problem (U, d) a point $F(U, d) \in U$ representing the agreed-upon utility profile.

We now ask: what are the natural axioms a bargaining solution should satisfy, and is there a unique one that does?

Axiom: Scale Invariance

Consider the transformation $(U, d) \mapsto (\bar{U}, \bar{d})$ such that

$$\bar{U} = \left\{ (\bar{u}_1, \bar{u}_2) \middle| \bar{u}_i = \alpha_i u_i + \beta_i \text{ for some } \alpha_i > 0, \quad (u_1, u_2) \in U \right\},$$
$$\bar{d}_i = \alpha_i d_i + \beta_i.$$

Then,

$$F_i(\bar{U}, \bar{d}) = \alpha_i F_i(U, d) + \beta_i.$$

Axiom: Efficiency

$\nexists u \in U$ such that $u \gg F(U, d)$.

Axiom: Symmetry

If U is symmetric (i.e., $(u_1, u_2) \in U \iff (u_2, u_1) \in U$) and $d_1 = d_2$, then

$$F_1(U, d) = F_2(U, d).$$

Axiom: Independence of Irrelevant Alternatives (IIA)

Consider (U, d) and $(\bar{U}, d)$ such that $U \subseteq \bar{U}$. If $F(\bar{U}, d) \in U$, then

$$F(\bar{U}, d) = F(U, d).$$

Theorem 3.3

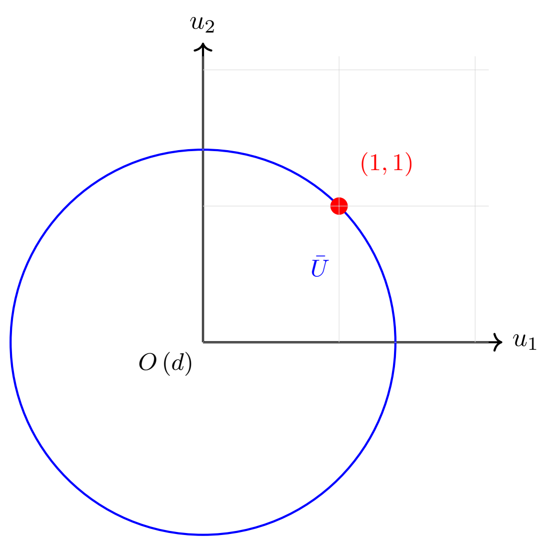
There is one and only one F that satisfies the above requirements. F is given by:

$$F^N(U, d) = \arg\max_{u \in U, u \geq d} (u_1 - d_1)(u_2 - d_2).$$

Proof for Theorem

Firstly, it is easy to check that F^N is well-defined in the sense that F^N satisfies the above requirements.

Let $(u_1^*, u_2^*) = F^N(U, d)$. We can always rescale the utilities such that $(u_1^*, u_2^*) \mapsto (1, 1)$ and $(d_1, d_2) \mapsto (0, 0)$, without loss of generality. Denote the new space as $(\bar{U}, \bar{d})$.



Claim

If we can show that $F = F^N$ for any F in the scaled space $(\bar{U}, \bar{d})$, then we must have $F = F^N$ for any (U, d).

Proof for Claim.

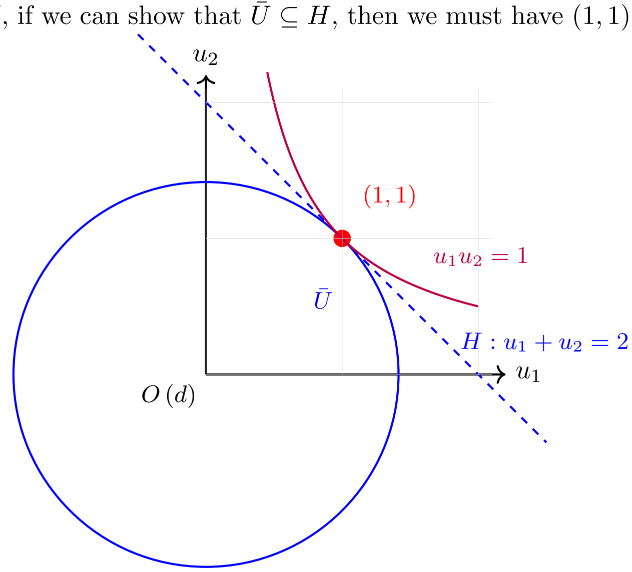
This is true by scale invariance.

Claim

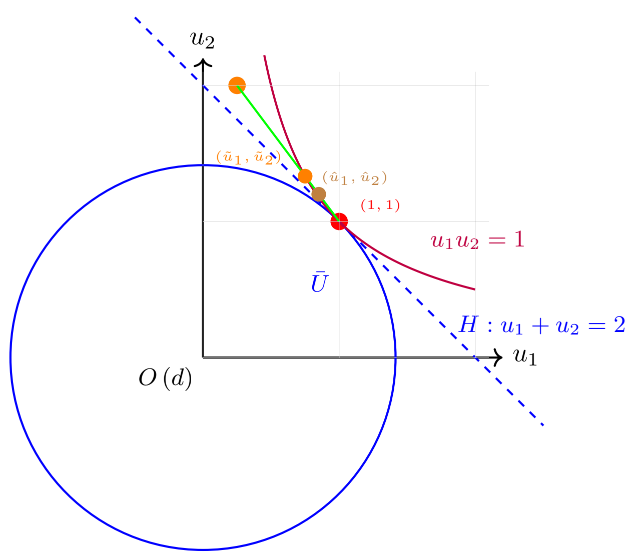
$F^N(\bar{U}, d) = (1, 1)$.

Proof for Claim.

Define the set $H = \{(u_1, u_2) \,|\, u_1 + u_2 \leq 2\}$. Note that the boundary of H is the tangent line to the hyperbola $u_1u_2 = 1$ through $(1,1)$ with slope -1. Since $(1,1) \in \bar{U}$ and $(1,1) \in H$, if we can show that $\bar{U} \subseteq H$, then we must have $(1,1)$ as the solution.



To show that $\bar{U} \subseteq H$, suppose for contradiction that $\bar{U}$ contains some point (u_1, u_2) such that $u_1 + u_2 > 2$. The line segment connecting $(1,1)$ and (u_1, u_2) must intersect the hyperbola $u_1u_2 = 1$ at some point $(\tilde{u}_1, \tilde{u}_2)$. So the line segment joining $(\tilde{u}_1, \tilde{u}_2)$ and $(1,1)$ must be above the hyperbola $u_1u_2 = 1$. Since $\bar{U}$ is convex, we can find a convex combination of (u_1, u_2) and $(1,1)$ that falls within the line segment joining $(\tilde{u}_1, \tilde{u}_2)$ and $(1,1)$, denoted by $(\hat{u}_1, \hat{u}_2)$ such that $(\hat{u}_1, \hat{u}_2) \in \bar{U}$ and $\hat{u}_1\hat{u}_2 > 1$. This contradicts the assumption that $F^N(\bar{U}, d) = (1,1)$.

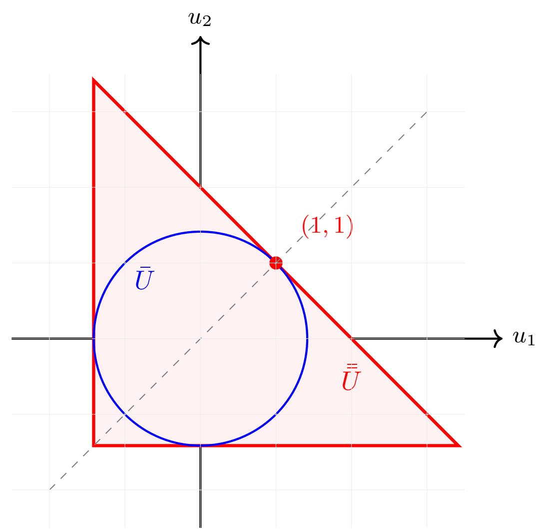


Claim

For any F, the solution must be $(1,1)$.

Proof for Claim.

We can always include the $\bar{U}$ using a symmetric right triangle whose hypotenuse is on the boundary of H and whose right angle sits on the line of $u_1 = u_2$. Denote the new triangle by $\bar{\bar{U}}$.



By efficiency of F, the solution must lie on the boundary of $\bar{\bar{U}}$. By symmetry of F, the solution must be $(1,1)$. Additionally, note that $\bar{U} \subseteq \bar{\bar{U}}$, and $F(\bar{\bar{U}}, \bar{d}) \in \bar{U}$. By IIA, we must have $F(\bar{\bar{U}}, \bar{d}) = F(\bar{U}, \bar{d})$. Therefore, $F(\bar{U}, \bar{d}) = (1,1)$.

From all the results above, we conclude that $F = F^N$ for any (U, d).

Remark.

This is actually a *normative* result in the sense that we require the bargaining solution to satisfy certain axioms. The Nash bargaining solution is the unique solution that satisfies these axioms. However, it is not necessarily a positive result in the sense that we do not have a game-theoretic model that can generate the Nash bargaining solution as an equilibrium outcome.

3.2 Dividing a Dollar

Let $X = [0,1]$. Suppose utility functions for the two players are $u_1 : X \to \mathbb{R}$ and $u_2 : X \to \mathbb{R}$, which are continuous, strictly increasing, and strictly concave. The problem is given by

$$\max_{x \in [0,1]} u_1(x) u_2(1-x).$$

The FOC is:
$$
\begin{aligned}
&u_1'(x^*)u_2(1-x^*) - u_1(x^*)u_2'(1-x^*) = 0 \\
\implies &\frac{u_1'(x^*)}{u_1(x^*)} = \frac{u_2'(1-x^*)}{u_2(1-x^*)}.
\end{aligned}
$$

Suppose in addition that Player 1 becomes more risk averse than Player 2:

$$
\begin{aligned}
&v_1 : X \to \mathbb{R}, \text{ where } v_1 = \phi(u_1), \quad \phi' > 0, \quad \phi'' < 0, \\
&u_2 : X \to \mathbb{R}.
\end{aligned}
$$

So the problem becomes:

$$\max_{x\in[0,1]} v_1(x)u_2(1-x) = \max_{x\in[0,1]} \phi(u_1(x))u_2(1-x).$$

The FOC is:

$$
\begin{aligned}
&\phi'(u_1(x^{**}))u_1'(x^{**})u_2(1-x^{**}) - \phi(u_1(x^{**}))u_2'(1-x^{**}) = 0 \\
\implies &\frac{\phi'(u_1(x^{**}))u_1'(x^{**})}{\phi(u_1(x^{**}))} = \frac{u_2'(1-x^{**})}{u_2(1-x^{**})}.
\end{aligned}
$$

Since ϕ is a concave function,

$$\phi'(t) < \frac{\phi(t)}{t} \implies \frac{\phi'(t)}{\phi(t)} < \frac{1}{t}.$$

Thus,
$$\frac{u_1'(x^*)}{u_1(x^*)} > \frac{\phi'(u_1(x^{**}))u_1'(x^{**})}{\phi(u_1(x^{**}))} = \frac{u_2'(1-x^{**})}{u_2(1-x^{**})}.$$

So in comparison,
$$\begin{cases} \frac{u_1'(x^*)}{u_1(x^*)} = \frac{u_2'(1-x^*)}{u_2(1-x^*)} \\ \frac{u_1'(x^{**})}{u_1(x^{**})} > \frac{u_2'(1-x^{**})}{u_2(1-x^{**})} \end{cases} \implies x^{**} < x^*.$$

3.3 Alternating Offer Model

Consider the problem of dividing a dollar: Two players will propose a division of a dollar in turns. Without loss of generality, suppose Player 1 proposes first. If Player 1 proposes to give x to Player 2, then Player 2 can either accept or reject. If Player 2 accepts, then the payoffs are $(1-x, x)$. If Player 2 rejects, then the game moves to the next period, where Player 2 proposes a division of a dollar and Player 1 can accept or reject. If Player 1 accepts, then the payoffs are $(x', 1-x')$. If Player 1 rejects, then the game continues to the next period, and so on. The game ends when one proposal is accepted. The discount factor for both players is $\delta \in (0,1)$.

Claim

Any split $(x, 1-x)$ where $x \in (0,1)$ is a NE outcome.

Proof for Claim.

This split can be supported by the strategies where Player 1 only accepts proposals that give them at least x, and Player 2 only accepts proposals that give them at least $1 - x$.

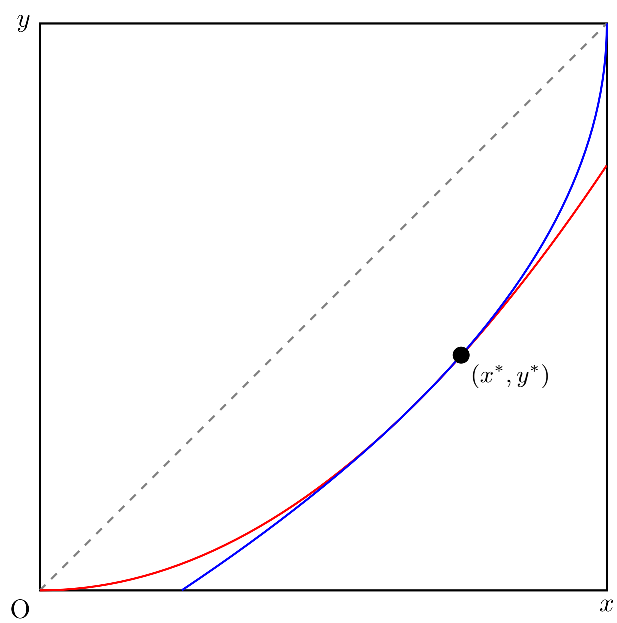
Theorem 3.4: Rubinstein

Let (x^*, y^*) be the solution to

$$u_1(y) = \delta u_1(x)$$
$$u_2(1-x) = \delta u_2(1-y)$$

Then there is a unique subgame perfect equilibrium in which in G_1, Player 1 asks for x^* and accepts iff $y \geq x^*$, and in G_2, Player 2 offers y^* and Player 1 accepts iff $x \leq y^*$.

Proof for Theorem



Let (x^*, y^*) be the solution to

$$u_1(y) = \delta u_1(x)$$
$$u_2(1-x) = \delta u_2(1-y)$$

Player 1 always offers x^*, and rejects Player 2's offer if $y < x^*$. Player 2 always offers y^*, and rejects Player 1's offer if $x > y^*$.

(z, t) is an outcome if a split of $(z, 1-z)$ is accepted in period t.

Define $v_1(z, t)$ as the dollar amount such that Player 1 is indifferent between receiving $v_1(z, t)$ today and receiving z in period t.

$$u_1(v_1(z,t)) = \delta^{t-1} u_1(z).$$

Define $v_2(z,t)$ as the dollar amount such that Player 2 is indifferent between giving Player 1 $v_2(z,t)$ today and giving Player 1 z in period t.

$$u_2(1-v_2(z,t)) = \delta^{t-1}u_2(1-z).$$

Define

$$M_i = \sup\{v_i(z,t) : (z,t) \text{ is a SPE of } G_i\},$$
$$m_i = \inf\{v_i(z,t) : (z,t) \text{ is a SPE of } G_i\},$$

where G_i is the subgame starting with Player i's proposal.

Claim

$$u_1(M_2) \leq \delta u_1(M_1).$$

Proof for Claim.

If Player 2 offers y such that $u_1(y) > \delta u_1(M_1)$, then Player 1 would accept the offer, because the right hand side $\delta u_1(M_1)$ is the most that Player 1 can get in the subsequent G_1. So Player 2 would never offer such y. Hence, we must have $u_1(M_2) \leq \delta u_1(M_1)$. ∎

Claim

$$u_1(M_2) \geq \delta u_1(M_1).$$

Proof for Claim.

If Player 2 offers y such that $u_1(y) < \delta u_1(M_1)$, then Player 1 will reject the offer, because by rejecting and entering G_1, Player 1 can get at most M_1 (with current payoff equivalent $\delta u_1(M_1)$). Therefore, any successful offer in G_2 must satisfy $u_1(y) \geq \delta u_1(M_1)$. Taking the supremum over all y in G_2 equilibria, we have $u_1(M_2) \geq \delta u_1(M_1)$. ∎

Claim

$$u_2(1-M_1) \geq \delta u_2(1-M_2).$$

Proof for Claim.

In G_1, if Player 1 demands more than $v_2(M_2, 2)$, then Player 2 will reject. This implies

$$M_1 \le v_2(M_2, 2).$$

And this implies

$$u_2(1 - M_1) \ge u_2(1 - v_2(M_2, 2)) = \delta u_2(1 - M_2).$$

Combining the three claims: $u_1(M_2) = \delta u_1(M_1)$ and $u_2(1 - M_1) \ge \delta u_2(1 - M_2)$. Together with $x^* \le M_1$ (by definition of supremum) and the fact that the system in (1) admits a unique solution, we obtain $(M_1, M_2) = (x^*, y^*)$.

An entirely symmetric argument applied to the infimum pair (m_1, m_2) yields $(m_1, m_2) = (x^*, y^*)$ as well: if Player 1's worst SPE outcome in G_1 were below x^*, Player 2 could mimic that outcome, and a parallel chain of inequalities (with sup and inf reversed) contradicts (1). Since sup and inf coincide, every SPE of G_1 delivers exactly x^* to Player 1, and every SPE of G_2 delivers exactly y^*. The proposed strategies form the unique SPE.

Proposition 3.5: Convergence to the Nash Bargaining Solution

Let $(z^*, 1 - z^*)$ be the Nash Bargaining Solution to $\max_{0 \le z \le 1} u_1(z)\, u_2(1 - z)$. Then as $\delta \to 1$, both $x^*(\delta)$ and $y^*(\delta)$ converge to z^*.

Proof for Proposition.

From the Rubinstein system $u_1(y^*) = \delta u_1(x^*)$, the multiplier $\delta < 1$ forces $y^* < x^*$ for every fixed δ, and as $\delta \to 1$ the multiplier vanishes so $x^* - y^* \to 0$.

Multiply the two equations in (1):

$$u_1(y^*)\, u_2(1 - x^*) \;=\; \delta^2\, u_1(x^*)\, u_2(1 - y^*),$$

which after rearrangement gives

$$u_1(x^*)\, u_2(1 - x^*) \;=\; u_1(y^*)\, u_2(1 - y^*).$$

So the two payoff vectors $(u_1(x^*), u_2(1 - x^*))$ and $(u_1(y^*), u_2(1 - y^*))$ lie on the *same* hyperbola $u_1 \cdot u_2 = \text{const}$. They are both efficient (they exhaust the unit pie), and their distance shrinks to zero as $\delta \to 1$. The unique limit point on the efficient frontier of this hyperbola family is exactly the NBS, which by definition maximizes $u_1 \cdot u_2$.

Remark (Strategic Foundation of an Axiomatic Solution).

This is the celebrated **Nash program** at work. The NBS was originally introduced as the unique axiomatic answer to the bargaining problem (scale invariance, efficiency, symmetry, IIA). The Rubinstein game gives it a *strategic* foundation: the unique SPE of an explicit non-cooperative bargaining protocol converges, as friction (impatience) vanishes, to the same solution the axioms picked out. The two perspectives—"what *should* a fair split be?" and "what *will* two patient bargainers actually agree on?"—turn out to coincide in the limit, lending each weight to the other.

Example (Rubinstein with Square-Root Utilities).

Let $u_1(x) = u_2(x) = \sqrt{x}$ and $\delta = 0.9$. The Rubinstein system reads

$$\sqrt{y^*} = \delta\sqrt{x^*}, \qquad \sqrt{1-x^*} = \delta\sqrt{1-y^*}.$$

Squaring both equations,

$$y^* = \delta^2 x^*, \qquad 1 - x^* = \delta^2(1-y^*).$$

Substituting the first into the second:

$$1 - x^* \ = \ \delta^2(1-\delta^2 x^*) \ = \ \delta^2 - \delta^4 x^*,$$

so $x^*(1-\delta^4) = 1-\delta^2$, giving

$$x^* \ = \ \frac{1-\delta^2}{1-\delta^4} \ = \ \frac{1}{1+\delta^2}.$$

Plugging $\delta = 0.9$ gives

$$x^* = \frac{1}{1+0.81} \ \approx \ 0.5525, \qquad y^* = \delta^2 x^* \ \approx \ 0.4475.$$

Player 1 (who proposes first) keeps about $1 - x^* \approx 0.4475$ for herself and offers $x^* \approx 0.5525$ to Player 2; Player 2, when proposing, would offer Player 1 only $y^* \approx 0.4475$. Notice the **first-mover advantage**: as $\delta \to 1$, both x^* and y^* tend to $1/2$ and the advantage vanishes; for finite δ, the proposer captures the strictly larger share.

Remark (The Symmetric-Discount, Symmetric-Utility Formula).

The example generalizes: whenever both players share the same utility function u and the same discount δ, the Rubinstein system always reduces to a single algebraic equation in x^*, with $y^* = u^{-1}(\delta\, u(x^*))$. The asymmetric case, treated next, requires reducing to a common discount factor by reweighting one player's utility—the clever transformation in the next section.

3.4 Unequal Discounting

Suppose now Player 1 has utility function w_1 and discount factor δ_1, and Player 2 has utility function w_2 and discount factor δ_2, with $\delta_1 < \delta_2$. We make the following transformation of the original problem:

- Player 2 has $u_2 = w_2$ and $\delta = \delta_2$.

- Player 1 has $u_1 = [w_1(x)]^{\frac{\ln \delta_2}{\ln \delta_1}}$ and $\delta = \delta_2$. (Note that $\frac{\ln \delta_2}{\ln \delta_1} < 1$ so that u_1 is concave, and is even "more concave" than w_1.)

  Without loss of generality, assume that

$$\begin{aligned}
&\delta_1^{t-1} w_1(z) > \delta_1^{t'-1} w_1(z') \\
\iff &\left[\delta_1^{t-1} w_1(z)\right]^{\frac{\ln \delta_2}{\ln \delta_1}} > \left[\delta_1^{t'-1} w_1(z')\right]^{\frac{\ln \delta_2}{\ln \delta_1}} \\
\iff &\frac{\ln \delta_2}{\ln \delta_1}\left[(t-1)\ln \delta_1 + \ln w_1(z)\right] > \frac{\ln \delta_2}{\ln \delta_1}\left[(t'-1)\ln \delta_1 + \ln w_1(z')\right] \\
\iff &(t-1)\ln \delta_2 + \frac{\ln \delta_2}{\ln \delta_1}\ln w_1(z) > (t'-1)\ln \delta_2 + \frac{\ln \delta_2}{\ln \delta_1}\ln w_1(z') \\
\iff &\ln \delta_2^{t-1} + \ln\left[w_1(z)\right]^{\frac{\ln \delta_2}{\ln \delta_1}} > \ln \delta_2^{t'-1} + \ln\left[w_1(z')\right]^{\frac{\ln \delta_2}{\ln \delta_1}} \\
\iff &\delta_2^{t-1} u_1(z) > \delta_2^{t'-1} u_1(z').
\end{aligned}$$

Remark (Chapter Summary).

Bargaining theory has two complementary strands. The *cooperative* strand axiomatizes a solution function $F(U, d)$ on the space of bargaining problems; the four Nash axioms (Pareto, symmetry, scale invariance, IIA) pin down the unique product-maximizing solution. The *non-cooperative* strand specifies a strategic protocol—Rubinstein's infinite-horizon alternating-offer game (Theorem 3.3)—and derives a unique SPE in stationary strategies. The connection between the two is the *Nash program*: as the time between offers shrinks (or, equivalently, as $\delta \to 1$), the unique SPE outcome of the Rubinstein game converges to the Nash bargaining solution of the underlying problem. Cooperative axioms and non-cooperative protocols thus pick out the same point. The chapter also showed how to handle asymmetric discount rates by reweighting one player's utility—a clever change of variables that reduces an asymmetric problem to a symmetric one and preserves the convergence result.

Part III

Auctions and Mechanism Design

Part IV

Matching

Part V

Information and Dynamic Games

Part VI

Problem Sets and Solutions

Part VII

Exams and Solutions