Game Theory: Midterm Review

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1 Games & Dominance

1.1 Strategic Game

1.1.1 Definition

Definition: A strategic game is denoted by $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$

- N: a set of **players**
- $\{S_i\}_{i \in N}$: a set of **strategies** for each player $i \in N$
- $\{u_i\}_{i \in N}$: a set of **payoff** functions, $\prod_{i=1} S_i \to R, \forall\ i \in N$

1.1.2 Strategy

- s_i, a strategy of player i
- S_i, a set of available strategies for player i
- $s := (s_i)_{i \in N}$, a strategy profile of all players

- $s_{-i} := (s_j)_{j \neq i}$, a strategy profile for all players except i
- $S := \prod_{i \in N} s_i$, the set of all (possible) strategy profiles
- $S_{-i} := \prod_{j \neq i} S_j$, the set of all (possible) strategy profiles for all players except i

1.1.3 Common Knowledge

An event E is common knowledge if

1. Everyone knows E.
2. Everyone knows that everyone knows E, and so on ad infinitum.

1.1.4 Hidden Assumptions

- All players are rational. ($\Leftrightarrow$ maximize utility over a "$\geq$").
- All players move at the same time.
- All components of the game are common knowledge.

1.2 Dominance

1.2.1 Strictly Dominant Strategy

- A strategy $s_i \in S_i$ strictly dominates $s_i' \in S_i \backslash \{s_i\}$ if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}), \forall \ s_{-i} \in S_{-i}$$

- A strategy $s_i \in S_i$ is a strictly dominant strategy for player i if s_i strictly dominates every strategy $s_i' \in S_i \backslash \{s_i\}$.
- A strategy profile $s^* = (s_i^*)_{i \in N}$ is a strict dominant-strategy equilibrium if s_i^* is a strictly dominant strategy for each $i \in N$.

Note

- **If a dominant-strategy equilibrium exists, it must be unique.**

- If one strategy strictly dominates another, it might also be dominated by some others.
- A rational player won't choose to play strictly dominated strategy at any time.

1.2.2 Weakly Dominant Strategy

- A strategy $s_i \in S_i$ weakly dominates $s_i' \in S_i \backslash \{s_i\}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall\ s_{-i} \in S_{-i}$$

  and for some $s_{-i}' \in S_{-i}$,

$$u_i(s_i, s_{-i}') > u_i(s_i', s_{-i}'), \forall\ s_i' \in S_i$$

- A strategy $s_i \in S_i$ is a strictly dominant strategy for player i if s_i weakly dominates every strategy $s_i' \in S_i \backslash \{s_i\}$.

- A strategy profile $s^* = (s_i^*)_{i \in N}$ is a weakly dominant-strategy equilibrium if s_i^* is a weakly dominant strategy for each $i \in N$.

Note

- A weakly dominant strategy must show its advantage over other strategies sometimes, though sometimes may not.
- Weak dominance is implied by strict dominance.
- *Weakly dominated* strategy is not the supplement to weakly dominant strategy, but to **strictly dominant** strategy.
- Dominance is assured under the assumption of common knowledge of rationality.
- Dominance, as a solution concept, has an advantage that its predictions are *independent* of players' beliefs about others' strategies. (**belief-free**)

1.3 IESDS

Iterated Elimination of Strictly Dominated Strategies

1. Let $G^0 = G = (N, \{S_i\}, \{u_i\}), S_i^0 = S_i \ \forall\ i \in N, S^0 = S$;
2. Let $S_{i,<}^0$ be the set of strictly dominated strategies for player i in game G^0, and define $S_i^1 = S_i^0 \backslash S_{i,<}^0$ for all $i \in N$;
3. If $S^1 := \prod_{i \in N} S_i^1 = S^0$, then done;
4. If not, let $G^1 = (N, \{S_i^1\}, \{u_i\})$, and construct the sets $S_{i,<}^1, S_i^2, S^2$ similarly;
5. Repeat the above procedures until $S^K = S^{K-1}$ for some $K \in \mathbb{N}$.

If s^* is a strictly dominant-strategy equilibrium for a game, then it uniquely survives the process of IESDS.

In any stage of IESDS, if s_i^* is a best response to s_{-i}^*, then as long as s_{-i} survives, so does s_i.(Not eliminated)

Note

- Dominance is a **strategy-by-strategy** concept, not a *cell-by-cell* or *Pareto-efficiency* concept!

- A strategy that is not strictly or weakly dominated in the original game may be eliminated in the reduced game.

- Strategies that survive the process of IESDS or IEWDS are **not** necessarily *weakly dominant* strategies! We can only say for sure that they *aren't* strictly or weakly dominated ones.

- **Strategies that are eliminated in the process of IESDS couldn't be part of NE, while those eliminated in IEWDS may be part of NE.**

  This is because weakly dominated strategies may perform equally best in some cases, and those cases may coincidentally constitutes NE, although weakly dominated strategies must perform worse than best sometimes. However, there stands no possibility for the strictly dominated one in the former situation.

- IESDS v.s. IEWDS
  - IESDS
    * IESDS does not always give sharp predictions, especially when the game involves many *weakly* dominant strategies.
    * IESDS may fail with games incorporating weakly dominant strategies.
    * **Order of elimination doesn't matter!**
  - IEWDS
    * Prediction power somehow improved, and partly fixes the problem with solutions of IESDS.
    * Order matters!

2 Nash Equilibirum

2.1 Best Response

2.1.1 Definition

A strategy $s_i \in S_i$ is player i's best response to $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall\ s_i' \in S_i$$

- Interpretation: if player i believes that his opponents will choose s_{-i}, then s_i is the rational choice for him.
- Existence: yes,under weak conditions. e.g.
  - if S_i is finite (by enumeration and comparison), or
  - if u_i is continuous and S_i is compact.
- Uniqueness: *not* guaranteed in general.

"Dominance" is a general *belief-free* concept, but "Best Response" is needed to focus more on rational choices under *specific* situations (say, $s_{-i} \in S_{-i}$).

- If s_i for player i is a best response to a certain $s_{-i} \in S_{-i}$, it doesn't necessarily mean that s_i is a weakly dominant strategy for player i, or that s_{-i} is a weakly dominated strategy.
- By extension, if s_i is a best response to all possible $s_{-i} \in S_{-i}$, then s_i is a weakly dominant strategy for player i.

2.1.2 Best Response & Dominant Strategy

1. **If s_i is a weakly dominant strategy for player i, then it is a best response to any $s_{-i} \in S_{-i}$.**
2. If s_i is a strictly dominated strategy for player i, then it cannot be a best response to any $s_{-i} \in S_{-i}$.

$\Leftrightarrow$ If s_i is a best response to any $s_{-i} \in S_{-i}$, then it is a weakly dominant strategy. $\Leftrightarrow$ Even with best response, players may still play weakly *dominated* strategies. $\Longrightarrow$ If s_i is a weakly *dominated* strategy for player i, then it can be a best response to *some* $s_{-i} \in S_{-i}$.

Proof

1. By definition, it holds.
2. Suppose s_i is a strictly dominated strategy. Hence, $\exists\ s_i' \in S_i, s.t.$

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}), \forall\ s_{-i} \in S_{-i}$$

Therefore, $\nexists\ s_{-i}, s.t.$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

By definition, s_i could not be a best response to any $s_{-i} \in S_{-i}$ in any case.

Note

- **Best response doesn't necessarily require the player to play the weakly or strictly dominant strategies**, but the most profitable. (Only IMpossible for strictly dominated strategies.)

2.1.3 Best Response & IESDS

Suppose S_i is finite $\forall\ i \in N$. If $s^* \in S$ uniquely survives the process of IESDS, then s_i^* is a best response to s_{-i}^*, $\forall\ i \in N$.

- Thus, the concept of best response would not contradict any sharp prediction obtained via the belief-free approach.

Proof

Suppose $s*$ uniquely survives the process of IESDS.

Suppose, in negation, that for some $i \in N$, s_i^* is not a best response to s_{-i}^*; also, S_i is finite, then

$$\begin{aligned} &\exists\ i \in N, \exists\ s_i' \in S_i \backslash \{s_i^*\}, \text{s.t.} \\ &s_i' \text{ is a best response to } s_{-i}^* \\ \implies &u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*) \end{aligned}$$

Pick any $\bar{s}_i \in BR_i(s_{-i}^*)$, (This announcement, fine if omitted)

Under the algorithm of IESDS, s_{-i}^* survives to the last and isn't eliminated. Since $\bar{s}_i$ (if announcement omitted, use s_i' as mentioned) is the best response to s_{-i}^*, then $(\bar{s}_i, s_{-i}^*)$ must have survived the process of IESDS. Note that $(\bar{s}_i, s_{-i}^*)$ is different from (s_i^*, s_{-i}^*), then (s_i^*, s_{-i}^*) is no longer the unique survivor after IESDS.

2.2 Nash Equilibrium

2.2.1 Definition

A strategy profile $s^* = (s^*)_{i \in N}$ is a Nash equilibrium if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$$

for all $s_i' \in S_i$ and every $i \in N$.

- No player would have a *strict* incentive to **unilaterally** deviate from s^*, meaning that it's **self-enforcing**.
- Not only that each player is best-responding to his beliefs, but also that they have **correct** beliefs.

2.2.2 Nash Equilibrium & Strategic Dominance

1. If $s^* \in S$ is a weakly dominant-strategy equilibrium, then s^* is a Nash equilibrium.
2. Suppose S_i is finite $\forall\ i \in N$. If $s^* \in S$ **uniquely** survives the process of IESES, then s_i^* is a Nash equilibrium.
3. Suppose S_i is finite $\forall\ i \in N$. If $s^* \in S$ is a Nash equilibrium, then s^* survives the process of IESDS.
4. In Nash equilibrium, players may still play dominated strategies.

Note

- Advantage of NE over IESDS: sharper predictions
- Advantage of NE over DS equilibrium: existence

Proof

2. If S_i is finite $\forall\ i \in N$ and $s^* \in S$ **uniquely** survives the process of IESES, then it's proved s_i^* is the best response to s_{-i}^* for all player $i \in N$. Then, it's proved (or by definition simply) $s^* = (s_i^*)_{i \in N}$ is a weakly dominant equilibrium. Lastly, obviously s^* is a NE.
3. Let $G^0 = N, \{S_i^0\}, \{u_i\}$, and suppose \$s^* \$ is the NE of G^0.

Consider the set of strictly dominated strategies, $S_{i,<}^0$ for each player i in G^0. Since s^* is the NE of G^0, then s^* won't be eliminated in the first round, and $s_i^* \notin S_{i,<}^0, \forall\ i \in N$.

Then consider G^1, s^* naturally belongs to S^1, and also naturally a NE of G^1.

The process goes on and on, with s^* never eliminated and being the survivor of IESDS.

3 Mixed-Strategy Nash Equilibrium

3.1 Definition

- Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a finite game

  i.e., a game where all N and $S_1, ..., S_N$ are finite sets.
- Let $S_i = \{s_{i1}, s_{i2}, ..., s_{im}\}$ be player i's strategy space.
- Let ΔS_i be the set of all probability distributions over S_i:

$$\Delta S_i = \{p : S_i \to [0, 1] \mid \sum_{k=1}^{m} p(_{ik}) = 1\}$$

- A mixed strategy of player i is then an element $\sigma_i \in \Delta S_i$.
- Expected payoffs

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i}) \cdot u_i(s_i, s_{-i})$$

  where $\sigma_{-i}(s_{-i}) = \prod_{j \neq i} \sigma_j(s_j)$.

Note

- A pure strategy is in the *support* of a mixed strategy if the mixed strategy places positive probability on that pure strategy.
- A pure strategy $s_i \in S_i$ is a special case of mixed strategies.

$$\sigma_i(s_i) = 1, \sigma(s_i') = 0 \ \forall \ s_i' \neq s_i$$

- A pure strategy of one player may never be a best response to any pure strategy of the other. However, it may be a best reponse to the other's mixed strategy, when probability falls in a certain region.
- A pure strategy can be dominated by a mixed strategy, even if the pure strategy is not dominated by any other pure strategy.

		P2 L	P2 R
P1	U	3 , 0	0 , 0
P1	M	0 , 0	3 , 0
P1	D	1 , 0	1 , 0

If P1 chooses to play U and M randomly, the payoff strictly dominates D as a pure strategy, even D isn't strictly dominated by neither U nor M.

3.2 Mixed-Strategy Nash Equilibrium

3.2.1 Definition

A mixed-strategy profile $(\sigma_1^*, ..., \sigma_n^*) \in \Sigma \equiv \prod_{i=1}^n \Delta S_j$ is a Mixed-Strategy Nash Equilibrium if

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(s_i, \sigma_{-i}^*),\ \forall\ s_i \in S_i, \forall\ i \in N$$

Note

- Pure-strategy NE is a special case of MSNE.
- In order to confirm a mixed-strategy profile σ^* is MSNE, only need to compare σ^* with all pure strategies! This holds because any mixed strategy is a convex combination of pure strategies.
- This version of definition greatly simplifies the process of confirming a MSNE, from all mixed strategies to only pure strategies as comparison objects. An more intuitive but complex definition of comparing mixed strategies is as follows:

  A mixed-strategy profile $(\sigma_1^*, ..., \sigma_n^*) \in \Sigma \equiv \prod_{i=1}^n \Delta S_j$ is a Mixed-Strategy Nash Equilibrium if

  $$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i', \sigma_{-i}^*),\ \forall\ \sigma_i' \in \Delta S_i, \forall\ i \in N$$

3.2.2 Indifference Principle

Let σ^* be a mixed-strategy NE, and $s_i, s_i' \in S_i$ be two pure strategy of player i. If

$$\sigma_i^*(s_i) > 0, \sigma_i^*(s_i') > 0$$

then,

$$U_i(s_i, \sigma_{-i}^*) = U_i(s_i', \sigma_{-i}^*)$$

Note

- In other words, **in any MSNE, the expected utility for each pure strategy in the support of the mixed strategy must be equal**. If not, obviously profitable deviation exists.
- **In any MSNE, the expected utility for each pure strategy not in the support of the mixed strategy must be less than or equal to the expected utility for each pure strategy in the support.** Specifically, suppose a player has A, B, C as three available pure strategies, if he only mixes A, B in a MSNE, then

$$u_i(A, \sigma_{-i}^*) = u_i(B, \sigma_{-i}^*) \geq u_i(C, \sigma_{-i}^*)$$

- By definition of MSNE, all the mixed strategies should be checked. From the indifference principle, only need to check **pure** strategies!
- Indifference Principle holds due to linearity and convexity.
- σ^* is MSNE iff

$$U_i(\sigma^*, \sigma_{-i}^*) \geq U_i(s_i, \sigma_{-i}^*), \forall\ s_i \in S_i, \forall\ i \in N$$

  - "$\Longrightarrow$": from the definition, pure strategy as special cases.
  - "$\Longleftarrow$": pure strategies can be mixed actively to obtain a certain mixed strategy.

3.2.3 Solution

Useful tips for finding MSNE:

- Consider a player i, take a subset S_i' of its strategies and assume that only these strategies are played by a strictly positive probability.
- Look for the other players' strategies that allow to satisfy
  1. The expected payoffs to play each one of the strategies in S_i' are equal to each other.

  $$E_i(s_j) = E_i(s_w), \forall\ s_j, s_w \in S_i'$$

  2. The expected payoffs to play each one of the strategies that are *not* in S_i' are not greater than the expected payoff of the strategies in S_i'.

  $$E_i(s_j) \leq E_i(s_w), \forall\ s_j \in S_i \backslash S_i', s_w \in S_i'$$

- Repeat this procedure for all possible strategies' subsets of player i.
- Repeat for all players.

Note

- Remember that NE is directly defined over **Best Response**. Taking each step carefully with "Best Response" will pay off!
- Indifference principle is built upon "Best Response". If you're to use it directly, mind the restriction of *positive* probability.
- For some of your strategies dominated by some other of yours (either pure or mixed scenario), purged! (If dominated by your alternative even before taking your opponent's mixed strategy, they deserve!)

3.3 NE Properties

3.3.1 Existence

Every finite strategic game has a Nash equilibrium.

Note

- The existence of NE can be interpreted as a fixed point.
- Infinite games do not guarantee the existence of NE.

3.3.2 Oddness Theorem

Almost all finite strategic games have a finite and **odd** number of Nash equilibria.

Note

- Inspired by number of fixed points of continuous functions.
- If a game seemingly has two (or even number of) NE, then double-check *Mixed*-Strategy Nash Equilibrium!

4 Justification & Refinements of Nash Equilibrium

4.1 Justification

- Objects of Choice
  - In the same way that people choose pure strategy as a best response.
  - Problem:
    * The games cannot capture the benefits of randomization. Suppose σ^* is a NE, if player i plays a mixed strategy σ_i, each s_i with $\sigma_i(s_i) > 0$ is a best response to σ^*. Then why bother to randomize?
    * Players are indifferent between different randomizations which have the same support. No reason to pick the precise randomization which supports the NE.
- A Steady State
  - Players act repeatedly and ignore any strategic links between plays.
  - Information about the frequencies that actions were taken in the past was received.

  - Each player uses these frequencies to form belief about the future behavior of the other players and pick his action.
  - Note:
    * **Only** Nash Equilibria has this property!
    * Alternatively, each player may represent a large population, whose behavior is described by the mixed strategy.
  - A MSNE is a steady state of the previous system in the sense that, if people stick with their mixing,
    * these frequencies remain fixed over time;
    * no one has an incentive to deviate unilaterally.
    * However, such steady state may not be stable.
- Frequency Summary of Predisturbed Game
  - A game is a frequently occurring situation in which the players' preferences are subject to small random variations.
  - In each occurrence of the situation, each player knows his own preferences, but not those of the other players.
  - A MSNE is a summary of the frequencies with which players choose their actions over time.

4.1.1 Steady State

4.1.1.1 Summarization If MSNE is viewed as a steady state, then players act **repeatedly** and ignore any strategic links between plays. Meanwhile, information about the frequencies that actions were taken in the past was received. Based on that, each player uses these frequencies to form belief about the future behavior of the other players and pick his action. Alternatively, each player may represent a large population, whose behavior is described by the mixed strategy.

A MSNE is a steady state of the previous system in the sense that, if people stick with their mixing, these frequencies remain **fixed** over time, and no one has an incentive to deviate *unilaterally*.

Therefore, to inspect the stability of MSNE, introduce a (transitory) **shock** which increases the likelihood of a pure strategy being played. And see, should we expect the system to 'return' to the NE? Answer is, some may, while some may not, and the steady state may not be stable. However, **ONLY NE has this property to be a steady state!**

		P2 A	P2 B
P1	A	1, 1	2, 2
	B	2, 2	1, 1

Suppose the probability to play A is p. Using the indifference principle, $p^* = \frac{1}{2}$ in MSNE.

Consider a probability shock to you, say $p < \frac{1}{2}$, then your opponent will be more inclined to play B, then you anticipate (or observe) this and play A more. Finally, you two players will pull the probability allocation back to MSNE state.

		P2 A	P2 B
P1	A	1, 1	0, 0
	B	0, 0	1, 1

Suppose the probability to play A is p. Using the indifference principle, again $p^* = \frac{1}{2}$ in MSNE.

Consider a probability shock to you, say $p < \frac{1}{2}$, then your opponent will be more inclined to play B, then you anticipate (or observe) this and play B more. You two will play B more and more, happier and happier, dragging the situation far away from MSNE.

4.1.1.2 Evolutionary Stability A mixed-strategy profile σ^* is **Evolutionary Stable** if

(1) $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \forall\ \sigma_i \in \Delta S_i, i \in N$

(2) if $u_i(\sigma_i, \sigma_{-i}^*) = u_i(\sigma_i^*, \sigma_{-i}^*)$, then $u_i(\sigma_i^*, \sigma_i) > u_i(\sigma_i, \sigma_i)$

Note

- A NE is evolutionary stable because it can "fight off" invaders (i.e., players that always player some other strategies).
- The definition expresses that, even if today you have a strategy σ_i that is an alternative best response to σ^*_{-i} (i.e., unilateral deviation from NE), if tomorrow I copy your deviation σ_i, none of us would be happy compared to σ^*_i in NE.

4.1.2 Pure Strategies in Predistrubed Game

A game is a frequently occurring situation in which the players' preferences are subject to small random variations. In each occurrence of the situation, each player knows his own preferences, but not those of the other players (say, of incomplete information). A MSNE is a summary of the frequencies with which players choose their actions over time.

Consider the following Stag Hunt Game, in its predisturbed version, as

		P2 A	P2 B
P1	A	$9+\varepsilon t_1$, $9+\varepsilon t_2$	0 , 5
	B	5 , 0	7 , 7

where $\varepsilon > 0$ is fixed and known, and $t_1, t_2 \sim U[0,1]$ are independent and privately known.

The idea is that this mixed strategy distribution can be generated by a pure-strategy NE in a "predisturbed game", and consider a "cutoff" strategy.

$$s_i^k(t_i) = \begin{cases} A & \text{if } t_i > k, \\ B & \text{if } t_i \leq k. \end{cases}$$

Plug the "cutoff" strategy back into the game and, $(s_1^k(.), s_2^k(.))$ is a NE iff

$$k = \frac{1}{2\varepsilon} \cdot (\varepsilon + \sqrt{-6\varepsilon + \varepsilon^2 + 121} - 11)$$

and $\lim_{\varepsilon \to 0} k(\varepsilon) = \frac{4}{11}$. In equilibrium, A and B are played by each player with probability $\frac{7}{11}$ and $\frac{4}{11}$, respectively.

4.2 Refinements

Refinements are additional conditions that allow us to rule out some "implausible" equilibria, in order to obtain sharper predictions.

- Evolutionary Stability
- Trembling-Hand Perfection
- Focal Points
- Risk/Payoff Dominance

4.2.1 Trembling-Hand Perfect Equilibrium

4.2.1.1 Definition

- A mixed strategy σ_i is completely mixed if $\sigma_i(s_i) > 0, \forall\ s_i \in S_i$.
- A strategy profile is a perfect equilibrium if there exists a sequence of completely mixed strategy profiles $\{\sigma^k\}_{k=1}^{+\infty}$, s.t.
  - $\lim_{k \to +\infty} \sigma^k = \sigma$
  - $u_i(\sigma_i, \sigma_{-i}^k) \geq u_i(s_i, \sigma_{-i}^k), \forall\ s_i \in S_i, i \in N, k = 1, 2, ...$

Note

The risk of trembling-hand enlightens players to think about possible NE that is strictly worse off and rule that out. Say, if others may tremble their hands between choices, you must take advantage of the potential good and avoid the potential harm. And if all players think through this logic, then the mutually harmful solution may prove to be implausible and be ruled out.

The perfect equilibrium may be claimed first, and then checked.

		P2	
		A	B
P1	A	1 , 1	0 , −3
	B	−3 , 0	0 , 0

Apparently, $(A, A), (B, B)$ are both pure-strategy Nash equilibria. It seems that (B, B) is strictly worse off than (A, A), and may even harmful for your own sake. However, under the assumption of rationality, (B, B) cannot simply be expelled. Even though, if little possibility is assigned to the Pareto efficient one, only a modicum, then not hard to prove both players will converge to it!

Claim that, (A, A) is a perfect equilibrium. Construct a corresponding sequence of completely mixed strategy profiles $\{\sigma^k\}_{k=1}^{+\infty}$, with $\sigma_i^k(A) = 1 - \frac{1}{k}, \sigma_i^k(B) = \frac{1}{k}$. Clearly, $\lim_{k\to+\infty} \sigma^k = \sigma = A$, and $u_i(B, \sigma_j^k) < u_i(\sigma, \sigma_j^k) = u_i(A, \sigma_j^k), \forall\ k = 1, 2, ...$ Proved.

4.2.1.2 Existence **Every finite strategic game has a perfect equilibrium.**

- For more detailed proof, see *Trembling-Hand Perfect Nash Equilibrium*.

4.2.1.3 Trembling-Hand Perfection & Weak Dominance

1. **A strategy profile is a perfect equilibrium only if it is a NE in which no player uses a weakly dominated strategy.**
2. A strategy profile is a perfect equilibrium in a finite two-player game iff it is a NE in which no player uses a weakly dominated strategy.

Proof

First prove that *every trembling-hand perfect Nash equilibrium is a Nash equilibrium.*

Let σ be a trembling-hand perfect Nash equilibrium. Pick any $\sigma_0 \in \Sigma_i$.

Note that $U_i(\sigma_i, \sigma_{-i}^n) - U_i(\sigma_i', \sigma_{-i}^n) \geq 0$ for all n. Then, continuity of U_i implies $U_i(\sigma_i, \sigma_{-i}) - U_i(\sigma_i', \sigma_{-i}) \geq 0$ and therefore $\sigma_i \in BR_i(\sigma_{-i})$ for all $i \in I$.

Then prove why no player would use a weakly dominated strategy under THP. Recall that the reason why the proposition make sense is that, even in NE, it's possible that players play weakly dominated strategies. The concept of perfect equilibrium comes up to deal with that situation.

In equilibrium, all pure strategies in the support of your mixed strategy must yield the same expected utility, and weakly dominated strategies sometimes produce a strictly worse payoff than other strategies. Hence, in a perfect equilibrium, your opponent mixes among all his strategies, and that sometimes will surely make your weakly dominated strategy worse than your weakly dominant strategies. So, it's rational for you not to play the weakly dominated strategy, and it won't be played in equilibrium, since you have a profitable deviation.

Note

- **A strategy profile that is a perfect equilibrium may not survive the process of IEWDS**, partly because a strategy that is not weakly dominated in original game may be weakly dominated in reduced game. See the following example:

	A	B	C
A	0,0	0,0	0,−2
B	0,0	1,1	−1,−2
C	−2,0	−2,−1	−2,−2

There are two pure strategy NEs: (A, A) and (B, B), with the former involving iterated weakly dominated strategies. And yet (A, A) is a THP equilibrium. To see this, consider the following sequence of totally mixed strategies

$$\sigma_i^k = (1 - \varepsilon^{-2k} - \varepsilon^{-k}, \varepsilon^{-2k}, \varepsilon^{-k}),\ i = 1, 2,\ \varepsilon > 2$$

which converges to $(1, 0, 0) = A$ as $k \to 0$. It is easy to verify that, for any

k, the best response of player $j \neq i$ to σ_i^k is A:

$$\begin{aligned} u_j(A, \sigma_k) &= 0 \\ u_j(B, \sigma_k) &= \varepsilon^{-2k} - \varepsilon^{-k} \\ u_j(C, \sigma_k) &= -2 \end{aligned}$$

Therefore, (A, A) is THP.

However. the game G' that survives iterated deletion of weakly dominated strategies does not contain the strategy A for either player. Hence, the THP equilibrium (A, A) of the original game G is not preserved in game G'.

4.2.2 Focal Points

A focal point is a solution that people tend to choose *by default* in the absence of communication.

Focal points often rely on things external to the description of the game.

4.2.3 Payoff/Risk Dominance

- A payoff-dominant equilibrium is any NE that Pareto dominates all others.
- A risk-dominant equilibrium is the NE that is least costly for hte players when they make mistakes.

  (s_1, s_2) risk-dominates (s_1', s_2') if

$$\begin{aligned} &(u_1(s_1', s_2) - u_1(s_1, s_2)) \cdot (u_2(s_1, s_2') - u_2(s_1, s_2)) \\ &\geq (u_1(s_1, s_2') - u_1(s_1', s_2')) \cdot (u_2(s_1', s_2) - u_2(s_1', s_2')) \end{aligned}$$

4.3 Zero-Sum Game

In finite, two-player, symmetric, zero-sum games, each player's equilibrium expected utility must be zero, and maxmin payoff

equals to minmax payoff.

Proof

- Finite game means the existence of Nash equilibrium.
- Symmetric game means one player can just copy another's strategy.
- Zero-sum means if copying, the payoff for each player must be zero.

5 Dynamic Games & Backward Induction

In the strategic games discussed before, all players move simultaneously, or at least they have no idea about what each other has done when they make their decisions.

5.1 Game Tree

5.1.1 Definition

A tree is a directed graph (X, E):

- X: set of nodes.
- $E \subset X \times X$: set of edges.
- $(x, y) \in E$: node x precedes node y.
- $r \in X$: root $(r, x) \in E, \forall\ x \in X \backslash \{r\}$.
- E is transitive and asymmetric.
  - Transitive: $(x, y), (y, z) \in E \Longrightarrow (x, z) \in E$;
  - Asymmetric: $(x, y) \in E \Longrightarrow (y, x) \notin E$.

Note

- (x, y) indicates a **direction** from x to y, and that is *irreversible.*
- (x, y) corresponds to a path, which may contain *two or more* nodes (i.e., one or more edges).

5.1.2 Properties

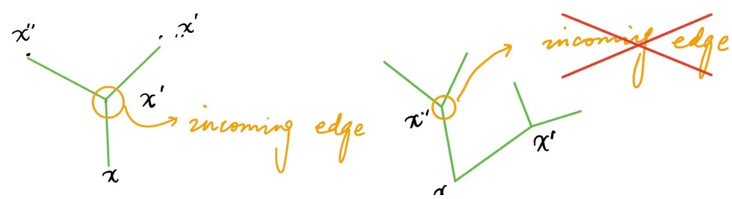
- Every node $x \in X \backslash \{r\}$ has one *incoming* edge.

$$\exists\ (x', x) \in E, s.t.$$
$$(x'', x) \in E, x'' \neq x' \Longrightarrow (x'', x') \in E$$

- Each node can be reached through a **unique** path.

Note

- The definition of the incoming edge guarantees that there's a path to every node except the root.
- Incoming edge itself determines that the path is unique.



5.2 Extensive-Form Games

5.2.1 Definition

An extensive-form game Γ consists of

- A set of players: $N = \{1, ..., n\}$;
- A tree (X, E);
- Action sets: $A(x)$ for all $x \in X$;

  Action sets are labels of outgoing edges (or, immediate successors).

- Payoff: $u_i : Z \to \mathbb{R}$, where $Z \equiv \{z \in X \mid \nexists\ x, (z, x) \in E\}$;

  Z is the set of all terminal nodes.

- A player map $I : X\backslash Z \to N$ that specifies the order of moves;

  $I(x)$, not so strictly speaking, marks who are to make decisions at node x.

- An information partition: $\mathcal{H} \equiv \{H^k\}_{k\in K}$ of $X\backslash Z$.

Notations

- $\mathcal{H}_i \equiv \{H^k : I(x) = i, \forall\ x \in H^k\}$, collection of all information sets at which player i plays.
- $A_i(H_i) \equiv A(x), \forall\ x \in H_i \in \mathcal{H}_i$, collection of all actions that player i can take at H_i.
- $A_i \equiv \cup_{H_i\in\mathcal{H}_i} A_i(H_i)$, collection of all the actions that player i may have access to.

Note

- The three notations above aim to define strategies in extensive-form games, in the sense that information sets classify the non-terminal nodes, and actions are better defined upon information sets than nodes.

5.2.2 Information Sets

An information set H^k is a collection of nodes s.t.

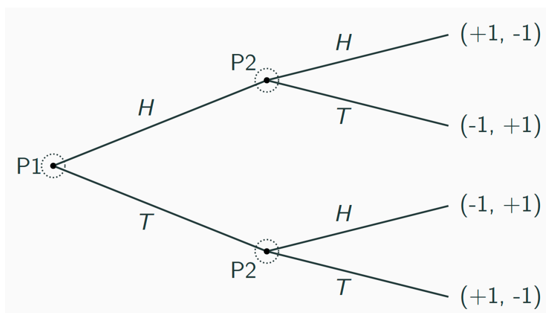
$$x, x' \in H^k \Longrightarrow I(x) = I(x'), \text{and } A(x) = A(x')$$

A game of complete information in which every information set is a singleton is called a game of perfect information.

Note

- An information partition allocates each non-terminal node of the game tree to an information set.

- In its definition, "$I(x) = I(x'), \text{and } A(x) = A(x')$" means classifies the states in the same stage into one information set (i.e., same player to make decisions, same decisions to choose).
- Information set captures the idea that a player may not always know which node he is at.
- The following two examples may help illustrate the concept of information sets and its power of classification (applied to actions, covered later).



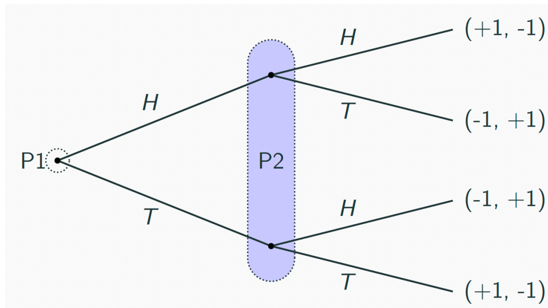
The root, the middle up and down points are labelled as x_1, x_2, x_3 respectively. Same in the next game.

The game above has 3 information sets. The always best way to find out the information sets is to figure out the action set and player map of each node, and then make classifications.

$$
\begin{aligned}
&A(x_1) = \{H, T\},\ I(x_1) = 1 \\
&A(x_2) = \{H, T\},\ I(x_2) = 2 \\
&A(x_3) = \{H, T\},\ I(x_3) = 2 \\
\Longrightarrow\ &H^1 = \{x_1\}, H^2 = \{x_2\}, H^3 = \{x_3\} \\
&H = \{H^1, H^2, H^3\}
\end{aligned}
$$

Note that although $A(x_2) = \{H, T\},\ I(x_2) = 2;\ A(x_3) = \{H, T\},\ I(x_3) = 2$ shows the "same" information, they have essentially different meanings.

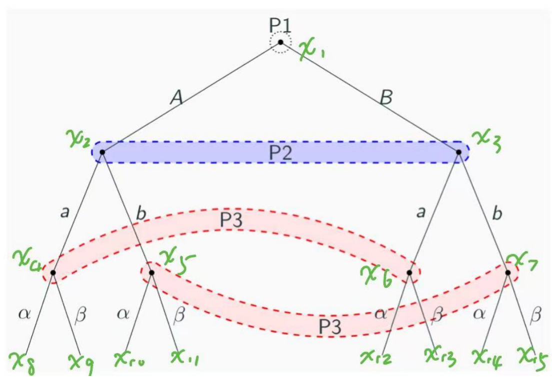
In this game, a distinct trait is that all players know exactly "where" he is at. But that's not the case in the following game.



P2 won't know where he's at after P1 made the choice. The information partition here becomes

$$H = \{H^1, H^2\} \Longleftarrow H^1 = \{x_1\},\ H^2 = \{x_2, x_3\}$$

One more example,



$$A(x_1) = \{A, B\}, A(x_2) = A(x_3) = \{a, b\}, \ldots$$
$$H^1 = \{x_1\}, H^2 = \{x_2, x_3\}, H^3 = \{x_4, x_6\}, H^4 = \{x_5, x_7\}$$

In the game, P3 knows P2's choice, but knows nothing about P1's.

5.3 Zermolo's Theorem

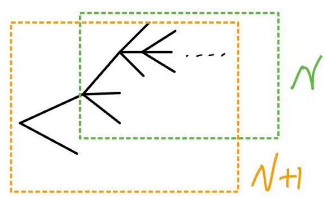
- **Every finite game of perfect information has a backward induction solution.**

- Moreover, if no two payoffs are the same for any player, then there is a **unique** backward induction solution.

Note

- Zermolo's theorem can be proved by induction.

  Obviously, this holds for one-stage game. Suppose this holds for N-stage game, then for a $N+1$-stage game, that can be taken apart into a N-stage game and a reduced one-stage game.



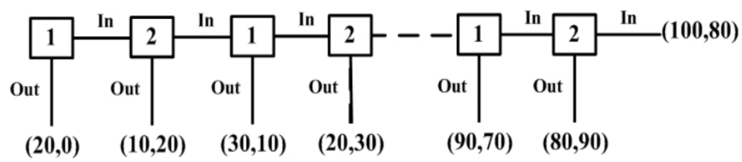
- Similar to finite strategic-form game's NE, always possible to solve extensive-form game via comparison. Similar logic still holds, and the process of induction guarantees its eligibility and existence of solution.
- Pratically, backward induction rules out empty threats (more detailed later), while rigid commitments make threats credible.
- However, in imperfect information games, backward induction may fail to give a solution.(Rigorously speaking, the process of solving the incomplete-information subgame isn't the same as backward induction, but has quite similar elements.)

5.4 Sequential Rationality

A player exhibits sequential rationality if he maximizes his expected payoff, conditional on every information set at which he has the move.

- Sequential rationality requires each player to optimize even at those information set that the player does not believe will be reached.
- *Even if players made mistakes in the past, they will play rationally in the future.* Some "mistakes" deviating from so-called rationality may

imply precious information or others' expectations. In this case, the late-movers are strongly assumed not to "learn" from that mistake, but only focus on now and rationality. For example,



Using backward induction, P1 to choose "Out" initially is the solution. However, that solution "incurs" great losses to both players. If P1 takes his audacity to risk "In", this must indicate he has the expectation of earning a higher payoff. Then, it's P2 to choose. Though under rationality, choosing "Out" is somehow right; if P2 digests P1's message, and he is somehow willing to cooperate for better, the situation may change on and on! However, this scenario is assumed to, anyway, out of range.

6 Subgame Perfect Nash Equilibrium

6.1 Redefinition

Bear in mind the differences between simultaneous games and dynamic games. Some well-defined concepts may be inappropriate to be transplanted directly. Therefore, redefining relative concepts are the primary issue.

6.1.1 Pure Strategy

A pure strategy for player i is a mapping

$$s_i : \mathcal{H}_i \to A_i$$

such that $s_i(H_i) \in A_i(H_i)$ for every information set $H_j \in \mathcal{H}_i$.

Note

- "$s_i(H_i) \in A_i(H_i)$" means your choice must itself be available.
- A pure strategy is a *complete contingent plan* of actions at each information set.
- Differentiate between **actions** and **strategies**. An action is a move you make at a stage of a game, while a strategy is a plan of actions conditional on any possible contingency.

6.1.2 Strategic-Form Representation

Let $\Gamma = (N, (X, E), A(.), \{u_i\}, I, \mathcal{H})$ be an extensive-form game. The strategic-form game associated with Γ is given by

$$G = (N, \{S_i\}_{i \in N}, \{\hat{u}_i\}_{i \in N})$$

where S_i is the set of all pure-strategy mappings of player i, and

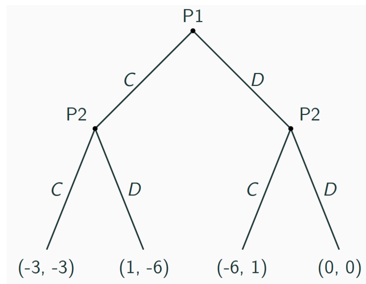
$$\hat{u}_i(s) = u_i(z_s), \forall\ s \in \prod_{i \in N} S_i$$

with z_s being the terminal node that will be reached when the strategy profile s is played.

Note

- The definition "breaks" the innate barriers between static games and dynamic games.
- Represented as a strategic-form game, especially "$G = (N, \{S_i\}_{i \in N}, \{\hat{u}_i\}_{i \in N})$" requires all the players to choose the plan at the very beginning.
- Utility is measured through the eventual-ending node, even though each step in the initial plan may not directly bring the corresponding utility.

The following are two examples, a game of complete and incomplete information respectively.



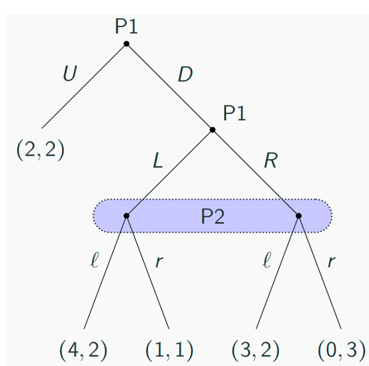
In strategic-form representation:

$$-N = \{1, 2\}$$
$$-S_1 = \{C, D\}, S_2 = \{CC, CD, DC, DD\}$$
$$-u_1(C, CC) = u_2(C, CC) = -3;$$
$$u_1(C, DC) = 1, u_2(C, DC) = -6; ...$$

Note

- Pure strategy is expressed like a set of contingent plan, so in this example, pure strategy DC means that, player 2 will choose D conditional on player 1's choice of C, and that player 2 will choose C conditional on player 1' s choice of D.
- In a game of complete information, nodes at the same stage corresponds to different information sets. Therefore, the pure strategy, the contingent plan, should be "allocated" to each possible state.

Then, move on to an incomplete information game.



In strategic-form representation:

$$-N = \{1, 2\}$$
$$-S_1 = \{UL, UR, DL, DR\}, S_2 = \{l, r\}$$
$$-u_1(U, l) = u_1(U, r) = u_2(U, l) = u_2(U, r) = 2;$$
$$u_1(DL, r) = 1, u_2(DL, r) = 1; ...$$

Note

- Weird as it seems at first glance, the strategic-form representation strictly follows the definition. Understand thoroughly and it will help a lot.
- **For a specific player, the number of elements in his pure-strategy set should equates the number of information sets for him.**
- All players make a complete plan before the game starts. And their utility follows every combination of the plan, even though in some combined plan, the game will end at the earlier stage, like player 1's choosing U initially.
- For incomplete information, player 2 cannot differentiate where he's at when he is to make an action. Hence, the contingent plan can only be built on player 1's first-stage choice, instead of the second-stage.

6.1.3 Mixed & Behavior Strategy

6.1.3.1 Mixed Strategy A mixed strategy for player i is a probability distribution $\sigma_i \in \Delta S_i$.

- The player selects a plan of actions randomly **before the game starts** and then follows that particular plan.
- However, when in a game tree, more natural to think of a player randomizing strategies conditional on the node he is at.

6.1.3.2 Behavior Strategy A behavior strategy for player i is a mapping

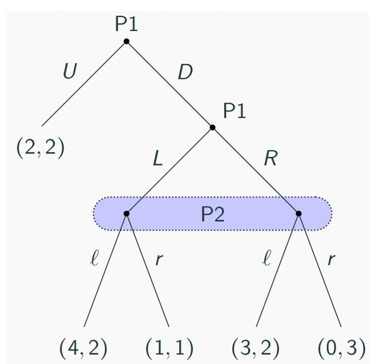
$$b_i : \mathcal{H}_i \to \Delta A_i$$

such that $b_i(H_i) \in \Delta A_i(H_i)$ for every information set $H_i \in \mathcal{H}_i$.

Note

- Behavior strategy is a "**conditional**" concept, compared to mixed strategy as a pre-determined and unconditional concept.
- Behavior strategy captures the idea that the player mixes his strategies whenever he has to, which is more natural to think about case-by-case choice.

6.1.3.3 Equivalence of Mixed & Behavior Strategy A mixed strategy σ_i and a behavior strategy b_i are equivalent if, for every σ_{-i} (or b_{-i}), both of them led to the *same distribution of outcomes* (i.e., the modes at which the game terminates).

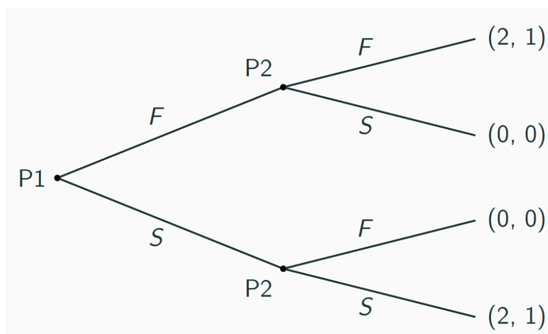


Note

- This means, given what others have chosen (set the path for you), choosing either mixed strategy or behavior strategy for the remaining path both work fine (with same probabilities).
- Remember, the number of behavior strategies equates the number of information sets for the player, and the number of pure strategies

equals to all possible combination of contingencies.

- Mixed strategy is more like a *commitment* made at the outset, while behavior strategy is more like *contingent plans*. They aren't necessarily equivalent, but equivalent under some conditions.
- Neither mixed strategy nor behavior strategy is "stronger" or implied by the other.
- Moreover, the equivalence of mixed and behavior strategy frees us to jump from a exponential-growing-dimensional mixed-strategy problem to a just-linear-growing-dimensional behavior-strategy one. Like the following example:



For mixed strategy, $\sigma_2 \in \Delta S_2$, where $S_2 = \{FF, FS, SF, SS\}$.

For behavior strategy, $b_i : \{F, S\} \to \Delta(\{F, S\})$. Let α, β denote the probability to choose F when in $\{F\}, \{S\}$ respectively.

Clearly, mixed strategy is a 3D problem here, while behavior strategy is only 2D.

Under equivalence, easy to get that

$$\alpha = p_{FF} + p_{FS}, \quad \beta = p_{SF} + p_{FF}$$

The process can be carried inversely, to construct a payoff-equivalent mixed strategy from a behavior one.

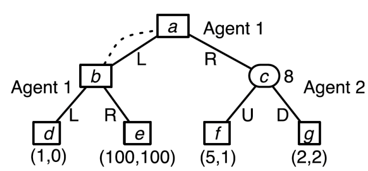
However, sometimes there's no payoff-equivalent strategy, if some players don't have perfect recall.

6.1.3.4 Perfect Recall

- In an imperfect-information game G, agent i has perfect recall if i *never forgets* anything he knew earlier. In particular, i remembers all his own moves.
- G is a game of perfect recall if every agent in G has perfect recall.
- **For every history in a game of perfect recall, no agent can be in the same information set more than once.**

Note

- For simultaneous games embedded in a game tree, there must be some players who don't know where they are at. But that isn't imperfect recall. Simply put, an agent without perfect recall is kind of perplexed about which **stage** he is at.
- An agent without perfect recall, the best way to think about this, is as if he encounters the same information set **twice** and make corresponding actions twice. If he takes mixed strategy, he kind of "commits" to that; if he takes behavior strategy, he considers it twice. Sometimes it seems that behavior strategy is "stronger" than mixed strategy due to drawbacks of commitments. The absent-minded driver is a good example. Here is the modified version.

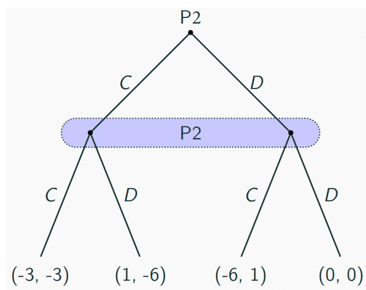


For agent 1, if he uses mixed strategy $\{(p, L), (1-p, R)\}$, then, if he chooses $L \;\rightarrow$ Game ends at d; if he chooses $R \;\rightarrow$ Game ends at either f or g. Interestingly, the game never ends at e.

If agent 1 uses behavior strategy $\{(q, L), (1-q, R)\}$ every time he is in information set $\{a, b\}$, then the game ends at e with probability $q(1-q)$, non-zero iff $q \neq 0, 1$; the game ends at d with probability q^2.

Apparently, the mixed strategy and behavior strategy won't reach an equivalence iff $q \neq 0, 1$.

- However, sometimes behavior strategy seems “stronger” than mixed strategies due to strength of commitments. Consider a absent-minded man to make two consecutive choices,



In stage 2, the player should forget the choice he has just made in the previous stage! In this case, mixed strategy plays a role of commitment device and assigns probabilities to each outcome with more “freedom”. In contrast, behavior strategy fall short of such “freedom”, and probabilities of each paired (C, D) are proportional. Therefore, a mixed strategy may not be replicated by a behavior strategy here.

6.1.3.5 Kuhn’s Theorem In every game in extensive form, if player i has *perfect recall*, then for every mixed strategy of i, there exists an **equivalent** behavior strategy.

Note

- Great relief to worries about existence of equivalent strategy!
- As long as no stage-perplexed players.
- Behavior strategy preferred under multi-stage problems.

6.1.4 Nash Equilibrium

A Nash equilibrium of an extensive-form game is defined as a Nash equilibrium of its *associated strategic-form game*.

- An extensive-form game has a NE whenever its associated strategic-form game has one.
- By Nash's theorem, every finite strategic-form game has a NE, and possibly in mixed strategies.

6.2 Subgame

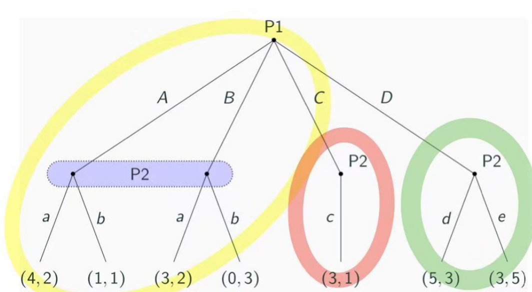
6.2.1 Definition

A subgame of an extensive-form game is any part of its game tree that meets the following criteria:

- It has an *initial* node that lies in a **singleton** information set.
- If a node is contained in the subgame, so are **all of the successors of that node**.
- If a node is contained in the subgame, so are all other **nodes in the same information** set of that node.

Note

- Pay special attention to embedded simultaneous games. See the example below.



The extensive-form game has 3 subgames, itself as the very first one, and two others are those circled in red and green. Note that the game circled in yellow doesn't follow the definition of subgame, since for the root, not all its successors are contained in the game.

6.2.2 Subgame Perfect Nash Equilibrium

6.2.2.1 Definition A strategy profile in an extensive-form game is a Subgame Perfect Nash Equilibrium if it contains a NE in every subgame.

6.2.2.2 Existence For every finite extensive-form game, there exists a SPNE.

6.3 An Algorithm for SPNE

1. Start from the **bottom** of the game tree.
2. For each subgame, find all the NE.
3. Plug in each NE and go back one stage to solve the higher level (reduced) subgame. If multiple NE, **discuss expectations** of the NE, either pure-strategy or mixed-strategy.
4. Repeat until the entire game is solved.

This algorithm for SPNE coincides with backward induction when the game is finite and of perfect information.

6.4 Backward v.s Forward Induction

- Forward induction is not a refinement of SPNE.
  - Central to the Forward Induction concept is that previous play tells you something about future play.
  - Subgames cannot be treated in isolation.
- Despite being intuitively plausible, formalizing notion of Forward Induction has proved tricky.

7 Midterm Exam 2022

7.1 Problems & Solutions

Problems and Solutions of Midterm Exam of 2022 are as follows.

- *Mid-term Exam, Game Theory, Spring 2022*
- *Mid-term Exam: Solution, Game Theory, Spring 2022*

7.2 Reminder

- Use directly stuffs in the slides! Otherwise, be cautious.
- Pay attention to **restrictions** to the values of variables.
- For the payoff function of a strategic-form game, fine to present in a payoff matrix.
- [Hint] is a bonus question. Not mandatory. If so difficult for you, just skip first and come back later.

7.3 Iterated Dominance

7.3.1 Best Response Function

- Best response function will give some hints to the follow-up question of undominated strategies, since the best response strategy cannot be strictly dominated.
- Best response is defined under "$\geq$", pay special attention to the case of "$=$".

7.3.2 Undominated Strategies

- Verify the undominated ones.

- **Verify the dominated ones.** (Necessary! Easy to be neglected!)

7.3.3 IESDS

- Find all strictly dominated strategies first and then eliminate.
- Mark the rounds of IESDS.
- Take advantage of **symmetry**!
- If asked to find all the NE, first conduct IESDS. Since if a strategy profile cannot survive the process of IESDS, it cannot be part of NE.

7.4 Nash Equilibrium

7.4.1 Pure Strategy

First find the pure strategy Nash equilibria. And use best response to accomplish that, because some dominated strategies will easily be eliminated.

Remember to rule out the possibility of the scenario that one player use a pure strategy, while the other use a mixed strategy. Recall that a best response to a pure strategy must be a pure strategy.

7.4.2 Mixed Strategy

Indifference Principle:

For $\sigma_i(A) > 0, \sigma_i(B) > 0, \sigma_i(C) = 0$, given other players' profile σ_{-i}, then

$$u_i(A, \sigma_{-i}) = u_i(B, \sigma_{-i}) \geq u_i(C, \sigma_{-i})$$

7.4.3 Extension

1. **Compare Trembling-Hand Perfect NE with IESDS, IEWDS, and dominated strategy.**

- A Trembling-Hand Perfect NE must be a NE.
- If a strategy cannot survive the process of IESDS, then it cannot be part of NE.
- If a strategy cannot survive the process of IEWDS, it may still be part of NE.
- **If a strategy cannot survive the process of IEWDS, it may still be part of Trembling-Hand Perfect NE.** (To be PROVED!)

2. More players, symmetric payoffs, binary actions. Find all symmetric Nash equilibria.

   $N = \{1, 2, ..., n\}$

   $\forall\ i,$ given others using $\sigma = (p, 1 - p)$

   write down the payoff for player $i, u_i(A, \sigma_j = \sigma, \forall j \neq i) = u_i(B, \sigma_j = \sigma, \forall j \neq i)$

7.5 Dynamic Games

7.5.1 Backward Induction

- List the whole procedure to avoid any possible mistakes!
- Mind the form of your result.
  - If asked to report the outcome, fine with plain descriptions.
  - If asked to report the SPNE, write rigorously in strategy-form. And don't leave out strategies in subgames, even if the game ends before the subgame.

7.5.2 Solution

7.5.2.1 Steps

- Solve the root game.
- Plug the NE back. If multiple NE, **discuss expectations** of the NE, either pure-strategy or mixed-strategy NE.

7.5.2.2 Report

- Fine to present the equilibrium in a game tree, and pin down all the actions in order.
- Better to present in behavior-strategy form for simplicity.

7.5.2.3 Extension Hopefully, the following two videos will help understand the Rubinstein's Alternating Offer Bargaining Game.

- *Two-Period Version*
- *Infinite-Period Version*

7.6 Important Concepts

- IESDS
- Best Response
- Indifference Principle
- **Refinement of NE: Trembling-Hand Perfect NE**
- **Evolutionary Stability**
- SPE