1 Homework 1

Problem 1. *(25 points)* The goal of this problem is to prove the following theorem regarding convergence of Fourier series.

Theorem. Let $\{a_n\}$ be a decreasing sequence of real numbers that converges to 0. Then the (Fourier) series $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise for all $x \in \mathbb{R}$ such that $\cos x \neq 1$.

a) *(10 points)* Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, let $b_1, \ldots, b_n \in \mathbb{R}$, and let $S_k = b_1 + \cdots + b_k$ for each k. Prove that

$$|a_1 b_1 + \cdots + a_n b_n| \leq a_1 \max(|S_1|, \ldots, |S_n|)$$

b) *(5 points)* Let $\{a_n\}$ be a decreasing sequence of real numbers converging to 0, and let $\{b_n\}$ be a sequence of real numbers for which $\sup_{n \in \mathbb{N}} |b_1 + \cdots + b_n| < \infty$. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

c) *(5 points)* Prove that

$$\sum_{n=1}^{N} \cos nx = \frac{\cos(N+1)x - \cos Nx - \cos x + 1}{2(\cos x - 1)}$$

for all $N \in \mathbb{N}$ and all $x \in \mathbb{R}$ for which $\cos x \neq 1$.

d) *(5 points)* Prove the *Theorem.*

Solutions.

a) Since $b_n = S_n - S_{n-1}$, we will have that

$$a_1 b_1 + \cdots + a_n b_n = a_1 S_1 + a_2(S_2 - S_1) + \cdots + a_n(S_n - S_{n-1}).$$

Rearrange the terms then use triangle inequality to get the desired result.

b) We need to use Cauchy's Criterion. Note that we can estimate

$$|a_n b_n + \cdots + a_m b_m|$$

by using part a) and the fact that a_n converges to 0 to finish the proof.

c) Use the identity

$$\cos(nx)\cos x = \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right).$$

d) Use part b) and c) with $b_n = \cos nx$ and note that x is fixed so $|b_1 + \cdots + b_n|$ is bounded when n varies.

□

Problem 2. *(10 points)* Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(\pi) = 0$, define real numbers $a_1, a_2 \ldots, a_n$ by the formula

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x) dx.$$

(*note that since f is continuous, the above integral is well-defined in any kind of integration*)
Show that the infinite sum $\sum_{n>0} a_n^2$ converges.

(*Hint*: *consider the expression* $\int_0^{\pi} \left(f(x) - \sum\limits_{k=1}^{n} a_k \sin(kx) \right)^2 dx$.)

Solutions. Consider the inequality

$$\int_0^{\pi} \left(f(x) - \sum_{k=1}^{n} a_k \sin(kx) \right)^2 dx \geq 0.$$

Expanding the left-hand side and noting that there are a lot of cancellations between the cross terms will yield

$$\int_0^{\pi} f^2(x) dx - \frac{\pi}{2} \sum_{k=1}^{n} a_k^2.$$

Since f is continuous on a closed interval, the integral of f^2 will be finite. This completes the proof. $\square$

Problem 3. (*10 points - Axler's 1B, 1*) Define $f : [0,1] \to \mathbb{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational} \\ 1/n & \text{if } a \text{ is rational and } n \text{ is the smallest positive} \\ & \text{integer such that } a = m/n \text{ for some integer } m. \end{cases}$$

Show that f is Riemann integrable and compute $\int_0^1 f$.

Solutions. Since the function is non-negative, we only need to show that the upper Riemann integral is zero. The key point in the proof is to note that there are only finitely many numbers in the interval $[0,1]$ are of the form m/n, where m and n are positive integers and $n < N$. Then we can construct a partition P of $[0,1]$ such that the sum of the lengths of the subintervals determined by P that contain a number of the form m/n, where m and n are positive integers and $n < N$, is at most $1/N$. Thus f is at most $1/N$ on the subintervals determined by P. Hence,

$$\mathcal{U}(f, P, [0,1]) \leq \frac{2}{N}.$$

Letting $N \to \infty$ yields the proof. ☐

Problem 4. *(5 points)* State (without proving) the Heine-Borel theorem and Bolzano-Weierstrass theorem. *There may be several versions of these theorems, state the ones that make most sense to you!*

Solutions. Google Search :).

□

2 Homework 2

Problem 1. *(30 points)*

i) *(5 points)* Prove that if A and B are subsets of $\mathbb{R}$ such that $\mu^*(B) = 0$ (B is a *null set*), then $\mu^*(A \cup B) = \mu^*(A)$.

ii) *(10 points)* Prove that if $A \subset \mathbb{R}$ and $t > 0$, then

$$\mu^*(A) = \mu^*(A \cap (-t,t)) + \mu^*(A \cap (\mathbb{R} \backslash (-t,t)).$$

iii) *(15 points)* Prove that $\mu^*(A) = \lim_{t \to \infty} \mu^*(A \cap (-t,t))$ for all $A \subset \mathbb{R}$.

Solutions.

i) Note that $\mu^*(A) \leq \mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$.

ii) Consider any collection of open intervals $\{I_k\}_{k \in \mathbb{Z}^+}$ whose union contains A. Note that $\{I_k \cap (-t,t)\}_{k \in \mathbb{Z}^+}$ is a collection of open intervals whose union contains A. Furthermore,

$$\{I_k \cap (-\infty, -t)\}_{k \in \mathbb{Z}^+}, \quad \{I_k \cap (t, \infty)\}_{k \in \mathbb{Z}^+}$$

will have union that contains $A \cap (\mathbb{R} \backslash (-t,t))$. Then by using definition of $\mu*$, we will get the desired result.

iii) Since $\mu^*(A \cap (-t,t))$ is an increasing function of t, we immediately have that

$$\mu^*(A) \geq \lim_{t \to \infty} \mu^*(A \cap (-t,t)).$$

For the other direction, note that

$$\begin{aligned}
\mu^*(A) &\leq \sum_{n=0}^{\infty} \mu^*(A \cap ((-n-1, n+1) \backslash (-n,n))) \\
&= \sum_{n=0}^{\infty} (\mu^*(A \cap (-n-1, n+1)) - \mu^*(A \cap (-n,n))) \quad \text{(by using part b)} \\
&= \lim_{n \to \infty} \mu^*(A \cap (-n-1, n+1)) \quad \text{(here n is an integer)} \\
&= \lim_{t \to \infty} \mu^*(A \cap (-t,t)) \quad \text{(here t is a real number.)}
\end{aligned}$$

☐

Problem 2. *(20 points, Axler's 1B, 5)* Show an example of a sequence of continuous real-valued functions $f_1, f_2, \ldots$ on $[0,1]$ and a continuous real-valued functions f on $[0,1]$ such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for each $x \in [0,1]$ but

$$\int_0^1 f(x)dx \neq \lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

Solutions. As long as $f_1, f_2, \ldots$ DOES NOT converge uniformly to f, it should work as an counterexample. For example, one can choose $f_k(x) = (k+1)(k+2)(x^k - x^{k+1})$ and $f(x) \equiv 0$ for $0 \leq x \leq 1$. There are many other examples. ☐

Problem 3. (*20 points*) Let $A \subset \mathbb{R}$ with $\mu^*(A) > 0$. Show that for every $\alpha \in (0,1)$ there exists an open interval I such that

$$\mu^*(A \cap I) \geq \alpha \mu^*(I).$$

(Hint: start with the definition of $\mu^*(A)$).

Solutions. The proof goes by contradiction. Suppose $\mu^*(A \cap I) < \alpha\mu^*(I)$ for any interval I. Consider an open covering $I_1, I_2, \ldots$ of A such that

$$\frac{1}{\alpha}\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(I_n) \quad \text{(this is possible because } 0 < \alpha < 1.)$$

We then have

$$\sum_{n=1}^{\infty} \mu^*(I_n) > \sum_{n=1}^{\infty} \frac{1}{\alpha}\mu^*(A \cap I_n) \geq \mu^*(A),$$

a contradiction. ☐

3 Homework 3

Problem 1. *(20 points, Axler's 2X, 10)* Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k).$$

Solutions. There are many examples. One example is $X = \mathbb{R}$, S is the Borel σ-algebra, μ is the outer measure and let $E_k = (k, \infty)$. $\square$

Problem 2. (*20 points*) Let $(X, \mathcal{S}, \mu)$ be a measure space such that there is $B \in \mathcal{S}$ such that $0 < \mu(B) < \infty$. Fix such a set B, and define a function $\mu_B : \mathcal{S} \to (-\infty, \infty)$ by the formula $\mu_B(A) := \mu(A \cap B)/\mu(B)$.

a) (*10 points*) Show that $(X, \mathcal{S}, \mu_B)$ is a measure space.

b) (*10 points*) Define the collection $S_B := \{A \cap B : A \in S\}$. Show that S_B is a σ-algebra on B.

(*This is how we define conditional probability in the language of measure theory.*)

Solutions. The proof is routine by just checking definitions of a measure and a σ-algebra.

$\square$

Problem 3. (*30 points*) Let (X, S, μ) be a measure space with $\mu(X) = 1$. Let $A_1, A_2, \ldots$ be a countable collection of sets in S such that $\mu(A_k) = 1$ for all $k > 0$. Prove that

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = 1.$$

(*Note that in this problem we* ***don't*** *assume A_k's are either decreasing or increasing.*)

Solutions. Show it for two sets A_1, A_2:

$$1 \geq \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cup A_2) \geq 1 + 1 - 1 = 1.$$

By induction, it will be true for any finite collection. Then consider a decreasing sequence $E_n = \bigcap_{k=1}^{n} A_k$ and apply continuity of measure to have the conclusion. ☐

4 Homework 4

Problem 1. (*40 points*) For this problem, to prove Lebesgue measurability of a set, you need to show that the set satisfies one of the *equivalent conditions* of Lebesgue measurability.

a) (*20 points*) Suppose $A \subset \mathbb{R}$ with finite outer measure $\mu^*(A) < \infty$. Prove that A is Lebesgue measurable **if and only if** for every $\varepsilon > 0$ there exists a set G that is the union of *finitely* many bounded open intervals such that

$$\mu^*(A \backslash G) + \mu^*(G \backslash A) \leq \varepsilon.$$

b) (*20 points*) Suppose $A \subset \mathbb{R}$ and $A \subset (b, c)$ for some $b < c$. Prove that A is Lebesgue measurable **if and only if**

$$\mu^*(A) + \mu^*((b,c) \backslash A) = c - b.$$

Solutions.

a) For forward direction, let $O = \cup_n I_n$ be a countable disjoint union of open intervals that cover A and such that $\sum_{n=1}^{\infty} \mu^*(I_n) = \mu^*(O) \leq \mu^*(A) + \varepsilon$. Since $\mu^*(A) < \infty$, there exists an integer $N > 0$ large enough such that

$$\mu^*(O \backslash \cup_{n=1}^{N} I_n) \leq \varepsilon/2.$$

Setting $G = \cup_{n=1}^{N} I_n$, this G should satisfy the conclusion.
For backward direction, note that we have neither the inclusion $A \subset G$ or $G \subset A$. Since $\mu^*(A \backslash G) < \varepsilon$, we can find an open set D containing $A \backslash G$ such that $\mu^*(D) < \varepsilon$ by definition. Then for open set $G \cup D$ which will contain A, the difference will be less than ε. Hence, A is measurable.

b) For forward direction, it is trivial from measurability.
For backward direction, let G and H be two open sets such that $A \subset G$ and $(b,c) \backslash A \subset H$ and

$$\mu * (G) \leq \mu^*(A) + \epsilon \quad \text{and} \quad \mu * (H) \leq \mu^*((b,c) \backslash A) + \epsilon.$$

We can show that

$$\mu^*(G \backslash A) \leq \mu^*((G \cup H) \backslash (b,c) \cup (G \cap H)).$$

The right-hand side can be computed explicitly since every set is measurable, which can be shown to be less than 2ε. Hence, A is measurable since $\mu^*(G \backslash A) \leq 2\varepsilon$.

$\square$

Problem 2. (*10 points*) Show that if a measurable set $A \subset [0,1]$ has positive Lebesgue measure $\mu(A) > 0$, then there are two elements x and y in A such that $|x-y|$ is an irrational number.

Solutions. By contradiction, if false, then A can be expressed as $A = \{x + q_n\}$ for some fixed $x \in A$ and a sequence of rational numbers $\{q_n\}$, which then implies A is countable and has measure zero. ☐

Problem 3. (*20 points*) Let A be a Lebesgue measurable subset of $\mathbb{R}$ with Lebesgue measure $\mu(A) > 0$. Define the set $A - A := \{x - y : x, y \in A\}$, the set of all of the differences of elements of A (for example, $[0,1] - [0,2] = [-2,1]$.) Show that $A - A$ contains an open symmetric interval around 0, i.e there exists $c > 0$ such that $(-c, c) \subset A - A$.
(*Hint: Use Problem 3 of HW2*)

Solutions. Without loss of generality, we can assume $\mu(A)$ to be finite since we can always consider $A \cap [-t, t]$ for some large t instead. By HW2, one can find an interval $I = (a, b)$ such that

$$\mu(A \cap I) \geq \frac{3}{4}\mu(I).$$

We will show that $A \cap I - A \cap I$ contains an open interval $(-c, c)$ instead (which will imply the conclusion of the problem). Let $B = A \cap I \subset I$. By contradiction, let us assume that there exists a sequence c_n that converges to 0 such that c_n doesn't belong to $B - B$ for all $n > 0$. Note that if a number $c \in \mathbb{R}$ doesn't belong to $B - B$ then $(B + c) \cap B = \emptyset$, which implies

$$0 = \mu((B + c) \cap B) = \mu(B + c) + \mu(B) = 2\mu(B).$$

We then have

$$2\mu(B) = \mu((B + c_n) \cap B) \leq \mu((I + c) \cap I = (b - a) + |c_n|.$$

However,

$$2\mu(B) \geq 2 \cdot \frac{3}{4}\mu(I) = \frac{3}{2}(b - a).$$

This leads to a contradiction since $\lim_{n\to\infty}((b - a) + |c_n|) = b - a$. ☐

5 Homework 5

Problem 1. (*20 points*) Give an example to show that Egorovs Theorem can fail if the measure of the whole space $\mu(X) = \infty$.

Solutions. Consider the case $X = \mathbb{R}$ and $f_n(x) = \chi_{[n,n+1)}$. ☐

Problem 2. (*20 points*) Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions on a measure space $(X, \mathcal{S}, \mu)$ and that

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x)| > 1/n\}) < \infty.$$

Prove that

$$\mu(\{x \in X : f_n(x) \text{ does not converge to } 0\}) = 0.$$

Solutions. We can show that

$$\{x \in X : f_n(x) \text{ does not converge to } 0\} \subset \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{x \in X : |f_n(x)| > 1/n\}.$$

Indeed, let $x \in X$ be such that $f_n(x)$ does not converge to 0. Therefore, there exists $\varepsilon > 0$ such that for all $N > 0$, there exists $n > N$ such that

$$|f_n(x)| > \varepsilon.$$

Choose $N > 1/\varepsilon$, then there exists an $n > N$ such that

$$|f_n(x)| > \varepsilon > \frac{1}{N} > \frac{1}{n}.$$

Note that we only have the LHS is a subset of the RHS. They are not the same set.

Then the measure of the left-hand side should be zero since the measure of the right-hand side is zero because it is less than the tail of the series in the hypothesis. ☐

Problem 3. (*20 points*) Let μ be the Lebesgue measure on $\mathbb{R}$. Let $b_1, b_2, \ldots$ be a sequence of real numbers. Define $f : \mathbb{R} \to [0, \infty]$ by :

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} \dfrac{1}{4^k |x - b_k|} & \text{if } x \notin \{b_1, b_2, \ldots\} \\ \infty & \text{if } x \in \{b_1, b_2, \ldots\}. \end{cases}$$

Show that $\mu(\{x \in \mathbb{R} : f(x) < 1\}) = \infty$.

(*Informally, this means $f(x) < 1$ (or, in fact, any number) on an infinite measure set.)*

Solutions. For each $k \geq 1$, consider the following set

$$E_k = \left[b_k - \frac{1}{2^k}, b_k + \frac{1}{2^k} \right].$$

Let E be the union of E_k's. Note that

$$\mu(E) \leq 2 \sum_{k=1}^{\infty} \frac{1}{2^k} = 2 < \infty.$$

Therefore, $\mu(\mathbb{R} \backslash E) = \infty$. Moreover, for $x \in \mathbb{R} \backslash E$, we have that

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{4^k |x - b_k|} < \sum_{k=1}^{\infty} \frac{2^k}{4^k} = 1.$$

We get the desired conclusion.

□