Real Analysis

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Chapter 0

Riemann Integration

0.1 Introduction: Fourier Series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\begin{cases} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \mathrm{d}x \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx \mathrm{d}x \end{cases}$$

In order to validate such results, the operations of $\int$ and $\sum$ for infinite terms should be interchangeable in computation order.

Justifying Fourier Analysis/Series under Riemann Integral is crazy. Alternatively, we would define as Lebesgue's Integration:

$$\int \sum^{\infty} = \sum^{\infty} \int$$

0.2 Riemann Integral

Definition 0.2.1: Partition

Suppose $a, b \in \mathbb{R}$ with $a < b$. A partition of $[a, b]$ is a finite list of the form $x_0, x_1, \cdots, x_n$, where $a = x_0 < x_1 < \cdots < x_n = b$.

A partition do not necessarily need to be an even partition.

Definition 0.2.2: Infimum and Supremum of a Function

If f is a real-valued function, A is a subset of the domain of f, then

$$\inf_A f := \inf \{f(x) : x \in A\}$$

$$\sup_A f := \sup \{f(x) : x \in A\}$$

Definition 0.2.3: Lower and Upper Riemann Sums

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P is a partition $x_0, x_1, \cdots, x_n$ of $[a, b]$.

- The lower Riemann sum $L(f, P, [a, b])$ is defined as

$$L(f, P, [a, b]) := \sum_{j=1}^{n} (x_j - x_{j-1}) \cdot \inf_{[x_{j-1}, x_j]} f$$

- The upper Riemann sum $U(f, P, [a, b])$ is defined as

$$U(f, P, [a, b]) := \sum_{j=1}^{n} (x_j - x_{j-1}) \cdot \sup_{[x_{j-1}, x_j]} f$$

The next result states that adjoining more points to a partition increases the lower Riemann sum and decreases the upper Riemann sum.

Proposition 0.2.4

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a, b]$ such that the listing defining P is a sublist of the list defining P'. Then

$$L(f, P, [a, b]) \leq L(f, P', [a, b]) \leq U(f, P', [a, b]) \leq U(f, P, [a, b]).$$

The following result states that if the function is fixed, then each lower Riemann sum is less than or equal to each upper Riemann sum.

Proposition 0.2.5: Lower Riemann Sums No More Than Upper Riemann Sums

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a, b]$. Then

$$L(f, P, [a, b]) \leq U(f, P', [a, b])$$

Now that we have been working with lower and upper Riemann sums, here we define the lower and upper Riemann integrals.

Definition 0.2.6: Lower and Upper Riemann Integrals

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. The *lower Riemann integral* $L(f, [a, b])$ and the *upper Riemann integral* $U(f, [a, b])$ of f are defined by

$$L(f, [a, b]) = \sup_{P} L(f, P, [a, b])$$
$$U(f, [a, b]) = \inf_{P} U(f, P, [a, b])$$

In the definition above, we take the supremum (over all partitions) of the lower Riemann sums because adjoining more points to a partition increases the lower Riemann sum and should provide a more accurate estimate of the area under the graph. Similarly, in the definition above, we take the infimum (over all partitions) of the upper Riemann sums because adjoining more points to a partition decreases the upper Riemann sum and should provide a more accurate estimate of the area under the graph.

Our intuition suggests that for a partition with only a small gap between consecutive points, the lower Riemann sum should be a bit less than the area under the graph, and the upper Riemann sum should be a bit more than the area under the graph. Intuitively, the lower and upper Riemann Integrals will converge to each other, or converge to a definite number, as long as the partition goes more and more finely. Both of them are approximations of the area under f on $[a, b]$.

Remark.

> By definition we immediately have $L(f, [a, b]) \leq U(f, [a, b])$. Instead of choosing between the lower Riemann integral and the upper Riemann integral, the standard procedure in Riemann integration is to consider only functions for which those two quantities are equal. When we take equality to this relationship, that is, $L(f, [a, b]) = U(f, [a, b])$, we call f (a bounded function on a closed bounded interval) *Riemann integrable*.

Definition 0.2.7: Riemann Integrable, Riemann Integral

A bounded function on a closed bounded interval is called *Riemann integrable* if its lower Riemann integral equals its upper Riemann integral. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the *Riemann integral* $\int_a^b f$ is defined by

$$\int_a^b f = L(f, [a, b]) = U(f, [a, b])$$

Proposition 0.2.8: Continuity and Riemann Integrability

Every continuous real-valued function on each closed bounded interval is Riemann integrable.

Proposition 0.2.9: Bounds on Riemann Integral

Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Then

$$(b-a) \inf_{[a,b]} f \leq \int_a^b f \leq (b-a) \sup_{[a,b]} f$$

0.3 Why is Riemann Integral NOT so good?

1. **Riemann integration does NOT handle functions with (too many) discontinuities.**

   Example.

   > Consider Dirichlet's function $f : [0,1] \to \mathbb{R}$:
   >
   > **Definition 0.3.1: Dirichlet Function**
   >
   > Dirichlet Function $f : [0,1] \to \mathbb{R}$ is defined as
   >
   > $$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$
   >
   > By definition, for $[a,b] \subseteq [0,1]$ with $a < b$, we have
   >
   > $$\begin{cases} \inf_{[a,b]} f = 0 \\ \sup_{[a,b]} f = 1 \end{cases} \implies \begin{cases} L(f,[0,1]) = 0 \\ U(f,[0,1]) = 1 \end{cases} \implies L(f,[0,1]) \neq U(f,[0,1])$$
   >
   > So we conclude that f is not Riemann integrable.
   >
   > Intuitively, since the set of rational numbers is countable and the set of irrational numbers is uncountable, $f(x)$ should be "not too" different from the function 0. In this way of reasoning, f should, in some sense, have integral 0. However, the Riemann integral of f is not defined.

2. **Riemann integration does not work well with unbounded functions.**

   Example.

   > Consider the following integration:
   >
   > $$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x$$
   >
   > Using the idea of improper integral, this gives an outcome of 2. However, say $x_0, x_1, \cdots, x_n$ is a partition of $[0,1]$, then $\sup_{[x_0,x_1]} f = \infty$. Thus if we tried to apply the definition of the upper Riemann sum to f, we would have $U(f,P,(0,1)) = \infty$ for every partition P of $[0,1]$.

3. **Riemann integration does NOT work well with pointwise limits.**

   Example.

   > Consider the Dirichlet's function. $\mathbb{Q}$ is countable. Let $r_1, r_2, \cdots$ be the sequence that indexes the rational numbers. For each positive integer k, we define f_k :

> $[0,1] \to \mathbb{R}$ such that
>
> $$f_k(x) = \begin{cases} 1, & \text{if } x \in \{r_1, \cdots, r_k\} \\ 0, & \text{otherwise} \end{cases}$$
>
> From Real analysis, we see that f_k is Riemann integrable, and $\int f_k = 0$ (which is a Riemann integral). However, when k goes to ∞, f_k converges to the Dirichlet's function, which is in contrast not Riemann integral.

Because analysis relies heavily upon limits, a good theory of integration should allow for interchange of limits and integrals, at least when the functions are appropri- ately bounded. Based on the three reasons, we hope to develop a new kind of integration that works well with the issues unsolved by the Riemann integration. We have to develop a new way to measure subsets of $\mathbb{R}$.