# Real Analysis

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# Contents

# Chapter 2

# Integration

## 2.1 Integration with Respect to a Measure

### 2.1.1 Integration of Nonnegative Functions

Through out this chapter we base our analysis on $(X, \mathcal{S}, \mu)$, which is a measure space.

**Definition 2.1.1: $\mathcal{S}$-Partition**

An *$\mathcal{S}$-partition* of $X$ is a finite collection $A_1, \cdots, A_m$ of disjoint sets in $\mathcal{S}$ such that

$$A_1 \cup \cdots \cup A_m = X.$$

**Definition 2.1.2: Lower Lebesgue Sum**

Suppose $f : X \to [0, \infty]$ is an $\mathcal{S}$-measurable function, and $P$ is an $\mathcal{S}$-partition $\{A_1, \cdots, A_m\}$ of $X$. The *lower Lebesgue sum* $L(f, P)$ is defined by

$$L(f, P) := \sum_{j=1}^{m} \mu(A_j) \inf_{A_j} f.$$

**Remark.**

- Notice that here we assume $f$ to be nonnegative, so $\inf_{A_j} f$ always exists.
- One convention is that, if any of the two terms is 0, the result is 0 and then neglected (regardless of whether or not the other term is $\infty$).

### Definition 2.1.3: Integral of a Nonnegative Function

The *integral of $f$ with respect to $\mu$*, denoted by $\int f \, \mathrm{d}u$, is defined by

$$\int f \, \mathrm{d}u = \sup_P \left\{ L\left(f, P\right) : P \text{ is an } \mathcal{S}\text{-partition of } X \right\}.$$

The following properties of integral are derived directly from the definition:

### Proposition 2.1.4: Properties of Integral

- $\int \chi_E \, \mathrm{d}\mu = \mu\left(E\right)$.
- Additivity: $\int \left(f + g\right) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$.
- Multiplication with a constant $c$: $\int cf \, \mathrm{d}\mu = c \cdot \int f \, \mathrm{d}\mu$.

In the remaining part of this section, we are going to examine each of the property and give proof for each.

### Proposition 2.1.5

If $E \in \mathcal{S}$, then

$$\int \chi_E \, \mathrm{d}\mu = \mu\left(E\right).$$

***Proof for Proposition.***

We separate the task to prove both $\int \chi_E \, \mathrm{d}\mu \leq \mu\left(E\right)$ and $\int \chi_E \, \mathrm{d}\mu \geq \mu\left(E\right)$.

Consider the partition $E, E^c$. Obviously $E \cup E^c = X$. From definition,

$$\begin{aligned}\int \chi_E \, \mathrm{d}\mu &\geq L\left(\chi_E, \{E, E^c\}\right) \\ &= \mu\left(E\right) \cdot \inf_E \chi_E + \mu\left(E^c\right) \cdot \inf_{E^c} \chi_E \\ &= \mu\left(E\right) + 0 \\ &= \mu\left(E\right)\end{aligned}$$

Then we prove the other direction of the inequality, i.e., $\int \chi_E \, \mathrm{d}\mu \leq \mu\left(E\right)$. It suffices to prove $L\left(\chi_E, P\right) \leq \mu\left(E\right)$.

Consider any $\mathcal{S}$-partition $P$, $A_1, \cdots, A_m$, such that $A_1 \cup \cdots \cup A_m = X$.

$$\begin{aligned}\mathcal{L}(\chi_E, P) &= \sum_{j=1}^{m} \mu(A_j) \inf_{A_j} \chi_E \\ &= \sum_{j:A_j \subseteq E} \mu(A_j) \cdot 1 \\ &= \mu\left(\bigcup_{j:A_j \subseteq E} A_j\right) \\ &\leq \mu(E)\end{aligned}$$

Taking supremium to both sides, we end with $\int \chi_E \, \mathrm{d}\mu \leq \mu(E)$, which when combined with the previous result completes the proof.

**Proposition 2.1.6**

Suppose $E_1, E_2, \cdots, E_n$ are disjoint sets in $\mathcal{S}$. $c_1, c_2, \cdots, c_n \in [0, \infty]$. Then

$$\int \left(\sum_{k=1}^{n} c_k \chi_{E_k}\right) \mathrm{d}\mu = \sum_{k=1}^{n} c_k \mu(E_k).$$

***Proof for Proposition.***

When one of the $c_1, \cdots, c_n$ is $\infty$, the equality is trivially true.

Let $E_{n+1} = X \backslash (E_1 \cup \cdots \cup E_n)$ and $c_{n+1} = 0$. Then $P = \{E_1, E_2, \cdots, E_n, E_{n+1}\}$ is an $\mathcal{S}$-partition of $X$. We have

$$\begin{aligned}\int \left(\sum_{k=1}^{n} c_k \chi_{E_k}\right) \mathrm{d}\mu &= \int \left(\sum_{k=1}^{n+1} c_k \chi_{E_k}\right) \mathrm{d}\mu \\ &\geq L\left(\sum_{k=1}^{n+1} c_k \chi_{E_k}, \{E_1, \cdots, E_{n+1}\}\right) \\ &= \sum_{k=1}^{n+1} c_k \mu(E_k) \\ &= \sum_{k=1}^{n} c_k \mu(E_k)\end{aligned}$$

Suppose $P$ is an $\mathcal{S}$-partition $\{A_1, \cdots, A_m\}$ of $X$. Then

$$
\begin{aligned}
L\left(\sum_{k=1}^{n} c_k \chi_{E_k}, P\right) &= \sum_{j=1}^{m} \mu\left(A_j\right) \inf_{A_j} \sum_{k=1}^{n} c_k \chi_{E_k} \\
&= \sum_{j=1}^{m} \mu\left(A_j\right) \min_{\{i: A_j \cap E_i \neq\}} c_i \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \mu\left(A_j \cap E_k\right) \min_{\{i: A_j \cap E_i \neq\}} c_i\right) \\
&\leq \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \mu\left(A_j \cap E_k\right) c_k\right) \\
&= \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu\left(A_i \cap E_k\right) \\
&= \sum_{k=1}^{n} c_k \mu\left(E_k\right)
\end{aligned}
$$

Note that the equality in the last line stands because of

$$
\begin{aligned}
A_1 \cup \cdots \cup A_m &= X \\
\sum_{j=1}^{m} \mu\left(A_j \cap E_k\right) &= \mu\left(\left(A_1 \cup \cdots \cup A_m\right) \cap E_k\right) \\
&= \mu\left(X \cap E_k\right) \\
&= \mu\left(E_k\right)
\end{aligned}
$$

**Corollary 2.1.7**

If $f(x) \leq g(x)$ for all $x \in X$, then

$$\int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu.$$

### 2.1.2 Monotone Convergence Theorem

**Definition 2.1.8: Integrals via Finite Simple Functions**

$$
\begin{aligned}
\int f \, \mathrm{d}\mu := \sup_{\{A_j, c_j\}} \Bigg\{ \sum_{j=1}^{m} c_j \mu\left(A_j\right) : & A_1, \cdots, A_m \text{ are disjoint sets of } X, \\
& c_1, \cdots, c_m \in [0, \infty], \\
& f(x) \geq \sum_{j=1}^{m} c_j \chi_{A_j}(x), \forall x \in X \Bigg\}
\end{aligned}
$$

## Theorem 2.1.9: Monotone Convergence Theorem

Let $0 \leq f_1 \leq f_2 \leq \cdots$ is an increasing sequence of $\mathcal{S}$-measurable functions. Define $f : X \to [0, \infty]$ as $f(x) := \lim_{k\to\infty} f_k(x)$. Then

$$\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

***Proof for Theorem***

Since $f_1 \leq f_2 \leq \cdots \leq f$,

$$\begin{aligned} &\int f_1 \, \mathrm{d}\mu \leq \int f_2 \, \mathrm{d}\mu \leq \cdots \leq \int f \, \mathrm{d}\mu \\ \implies &\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu \leq \int f \, \mathrm{d}\mu \end{aligned}$$

Now we are to focus on the other direction. Using the equivalent definition of integration, suppose $A_1, \cdots, A_m$ are disjoint sets of $X$, and $c_1, \cdots, c_m \in [0, \infty]$ such that

$$f(x) \geq \sum_{j=1}^{m} c_j \chi_{A_j}(x), \forall x \in X$$

Let $t \in (0, 1)$. For $k \in \mathbb{Z}_+$, let

$$E_k = \left\{ x \in X : f_k(x) \geq t \sum_{j=1}^{m} c_j \chi_{A_j}(x) \right\}$$

Then $E_1 \subseteq E_2 \subseteq \cdots$ is an increasing sequence whose union $\bigcup_{k=1}^{\infty} E_k = X$. Thus

$$\lim_{k\to\infty} \mu(A_j \cap E_k) = \mu\left( A_j \cap \left( \bigcup_{k=1}^{\infty} E_k \right) \right) = \mu(A_j), \forall j \in \{1, 2, \cdots, m\}$$

Moreover, from the construction of $E_k$,

$$f_k(x) \geq t \sum_{j=1}^{m} c_j \chi_{A_j \cap E_k}(x), \forall x \in X$$

Hence,

$$\int f_k \, \mathrm{d}\mu \geq t \sum_{j=1}^{m} c_j \mu(A_j \cap E_k)$$

Let $k \to \infty$, we have

$$\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu \geq t \sum_{j=1}^{m} c_j \mu(A_j)$$

Take $t \to 1$, we have

$$\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu \geq \sum_{j=1}^{m} c_j \mu(A_j).$$

*Why only take $t \in (0,1)$ instead of $t \in (0,1]$? Because if $t = 1$, we may not have $\bigcup_{k=1}^{\infty} E_k = X$.*

Next we will deploy Monotone Convergence Theorem to prove that the integration has the property of additivity, as is desired.

A simple function may have different representations, but the corresponding integrals are of the same value, which is formalized in the following proposition.

### Proposition 2.1.10: Integral-Type Sums for Simple Functions

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Suppose $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n \in [0, \infty)$, and $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n \subseteq \mathcal{S}$. Further suppose

$$\sum_{i=1}^{m} a_i \chi_{A_i}(x) = \sum_{j=1}^{n} b_j \chi_{B_j}(x), \forall x \in X.$$

Then

$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{j=1}^{n} b_j \mu(B_j).$$

**Remark.**

- Here it is not required that $\{a_i\}, \{b_i\}$ are distinct or $\{A_k\}, \{B_k\}$ are disjoint. From here we do not need to care about multiplicity about representations of simple functions.
- The standard representation for a simple function: $\sum_{i=1}^{m} a_i \chi_{A_i}$, where $a_1, a_2, \cdots, a_m$ are distinct and $A_1, A_2, \cdots, A_m$ are disjoint.

### Proposition 2.1.11: Additivity of Integration

If $f, g \geq 0$ are $\mathcal{S}$-measurable, then

$$\int (f+g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu.$$

***Proof for Proposition.***

Let $\{f_k\}$ converges pointwise to $f$ in an increasing order, so does $\{g_k\}$ to $g$. By MCT (applied to $\{f_k + g_k\}$),

$$\begin{aligned}
\int (f+g) \, \mathrm{d}\mu &= \lim_{k\to\infty} \int (f_k + g_k) \, \mathrm{d}\mu \\
&= \lim_{k\to\infty} \left( \int f_k \, \mathrm{d}\mu + \int g_k \, \mathrm{d}\mu \right) \\
&= \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu
\end{aligned}$$

where the equality in the second line stands because both $f_k$ and $g_k$ are simple functions, and the last one holds when applying MCT to $\{f_k\}$ and $\{g_k\}$ respectively.

### 2.1.3 Integration of Real-Valued Functions

**Definition 2.1.12: $f^+$, $f^-$**

Suppose $f : X \to [-\infty, \infty]$ is $\mathcal{S}$-measurable. Define $f^+ : X \to [0, \infty]$ and $f^- : X \to [0, \infty]$ by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

By definition, we see that

$$f(x) = f^+(x) - f^-(x)$$
$$|f|(x) = \left|f^+\right|(x) + \left|f^-\right|(x)$$

**Definition 2.1.13: Integral of a Real-Valued Function**

Suppose $f : X \to [-\infty, \infty]$ is an $\mathcal{S}$-measurable function such that at least one $\int f^+ \, \mathrm{d}\mu$ and $\int f^- \, \mathrm{d}\mu$ is finite. The integral of $f$ is

$$\int f \, \mathrm{d}\mu := \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu.$$

**Proposition 2.1.14: Properties of Integral**

Suppose $f : X \to [-\infty, \infty]$, and $g : X \to [-\infty, \infty]$.

- $\int cf \, \mathrm{d}\mu = c \int f \, \mathrm{d}\mu$, if $c \in \mathbb{R}$.
- $\int (f + g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$, if $\int |f| \, \mathrm{d}\mu < \infty$ and $\int |g| \, \mathrm{d}\mu < \infty$.
- $\int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu$, if $f(x) \leq g(x)$ for all $x \in X$.
- Triangle inequality: $\left|\int f \, \mathrm{d}\mu\right| \leq \int |f| \, \mathrm{d}\mu$.

## 2.2 Limits of Integrals & Integrals of Limits

### 2.2.1 Bounded Convergence Theorem

**Definition 2.2.1: Integration on a Subset**

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E \in \mathcal{S}$. Let $f : X \to [-\infty, \infty]$ be a $\mathcal{S}$-measurable function. The integral of $f$ over $E$ is defined by

$$\int_E f \, \mathrm{d}\mu := \int f \cdot \chi_E \, \mathrm{d}\mu.$$

The special case is that $\int_X f \, \mathrm{d}\mu = \int f \cdot \chi_X \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$. So when we refer to integration over the whole set $X$, we can simply write as $\int f \, \mathrm{d}\mu$ and omit the denotation of $X$.

**Proposition 2.2.2: Bounding an Integral**

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $E \in \mathcal{S}$, and $f : X \to [-\infty, \infty]$ is a function such that $\int_E f \, \mathrm{d}\mu$ is defined. Then

$$\left| \int_E f \, \mathrm{d}\mu \right| \leq \mu(E) \cdot \sup_E |f| .$$

***Proof for Proposition.***

$$\begin{aligned} \left| \int_E f \, \mathrm{d}\mu \right| &= \left| \int f \cdot \chi_E \, \mathrm{d}\mu \right| \\ &\leq \int |f| \cdot \chi_E \, \mathrm{d}\mu \\ &\leq \int \sup_E |f| \cdot \chi_E \, \mathrm{d}\mu \\ &= \mu(E) \cdot \sup_E |f| \end{aligned}$$

**Theorem 2.2.3: Bounded Convergence Theorem**

Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose a sequence of $\mathcal{S}$-measurable functions $f_1, f_2, \cdots$ converges pointwise to $f$. Suppose there exists $c \in (0, \infty)$ such that $|f_k(x)| \leq c$, for all $x \in X, k \geq 1$. Then

$$\lim_{k \to \infty} \int f_k \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

***Proof for Theorem***

Fix $\varepsilon > 0$. By Egorov's theorem, there exists $E \in \mathcal{S}$ such that

$$\mu(X \backslash E) \leq \frac{\varepsilon}{4c}$$

and $f_1, f_2, \cdots$ converges to $f$ uniformly on $E$. Then

$$\begin{aligned}
\left|\int f_k \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu\right| &= \left|\int_{X\setminus E} f_k \, \mathrm{d}\mu - \int_{X\setminus E} f \, \mathrm{d}\mu + \int_E f_k \, \mathrm{d}\mu - \int_E f \, \mathrm{d}\mu\right| \\
&\leq \left|\int_{X\setminus E} (f_k - f) \, \mathrm{d}\mu\right| + \left|\int_E (f_k - f) \, \mathrm{d}\mu\right| \\
&\leq \mu(X\setminus E) \sup_{X\setminus E} |f_k - f| + \mu(E) \sup_E |f_k - f| \\
&\leq \frac{\varepsilon}{4c} \cdot 2c + \mu(X) \cdot \frac{\varepsilon}{2(\mu(X)+1)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Hence, $\left|\int f_k \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu\right| \leq \varepsilon$ when $k$ is big enough. This implies that

$$\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

*Here it is in the last approximation of $\sup_E |f_k - f|$ that we used the definition (or, property) of uniform convergence. Egorov's theorem is a powerful tools in proofs that involves interchanging limits and integrals.*

### 2.2.2 Dominated Convergence Theorem

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. If $f, g : X \to [-\infty, \infty]$ are $\mathcal{S}$-measurable functions and $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$, then the definition of an integral implies that $\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu$ (or both are undefined). This is true because what happens on a set of measure 0 does not matter generally.

**Definition 2.2.4: $\mu$-Almost Every**

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. A set $E \in \mathcal{S}$ is said to contain $\mu$-almost every element of $X$ if

$$\mu(X\setminus E) = 0.$$

**Example.**

Almost every real number is irrational (with respect to the usual Lebesgue measure on $\mathbb{R}$) because $\mu^*(\mathbb{Q}) = 0$.

**Remark.**

- For two functions $f$ and $g$, if $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$, we usually say $f = g$ ($\mu$-)almost everywhere (a.e.).
- Theorems about integrals can almost always be relaxed so that the hypothesis apply

only almost everywhere instead of everywhere. One example is $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in X$ in Bounded Convergence Theorem.

### Proposition 2.2.5: Integrals on Small Sets are Small

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Suppose $g : X \to [0, \infty]$ is $\mathcal{S}$-measurable and $\int g \, \mathrm{d}\mu < \infty$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mu(B) < \delta$, then $\int_B g \, \mathrm{d}\mu < \varepsilon$.

***Proof for Proposition.***

Fix $\varepsilon > 0$. Construct $h : X \to [0, \infty)$ to be a simple $\mathcal{S}$-measurable function such that $0 \le h \le g$ and $\int g \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu \le \frac{\varepsilon}{2}$. (The existence can be show via the equivalent definition of Lebesgue integral.)

Let $H = \max\{h(x) : x \in X\}$ and let $\delta > 0$ be such that $H \cdot \delta \le \frac{\varepsilon}{2}$.

$$\begin{aligned}\int_B g \, \mathrm{d}\mu &= \left(\int_B g \, \mathrm{d}\mu - \int_B h \, \mathrm{d}\mu\right) + \int_B h \, \mathrm{d}\mu \\ &\le \frac{\varepsilon}{2} + \mu(B) \cdot \sup_B h \\ &\le \frac{\varepsilon}{2} + \delta \cdot H \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

■

### Proposition 2.2.6: Integrable Functions Live Mostly on Sets of Finite Measure

Let $g : X \to [0, \infty]$ and $\int g \, \mathrm{d}\mu < \infty$. Then for every $\varepsilon > 0$, there exists $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and $\int_{X \setminus E} g \, \mathrm{d}\mu \le \varepsilon$.

***Proof for Proposition.***

Let $\varepsilon > 0$. Let $P$ be an $\mathcal{S}$-partition $A_1, \cdots, A_m$ of $X$ such that

$$\int g \, \mathrm{d}\mu \le \varepsilon + L(g, P)$$

Let $E$ be the union of those $A_i \, (i = 1, \cdots, m)$ such that $\inf_{A_i} g > 0$. We can safely claim that $\mu(E) < \infty$; otherwise we would have $L(g, P) = \infty$, which contradicts the hypothesis that $\int g \, \mathrm{d}\mu < \infty$. Now

$$\begin{aligned}\int_{X \setminus E} g \, \mathrm{d}\mu &= \int g \, \mathrm{d}\mu - \int \chi_E \cdot g \, \mathrm{d}\mu \\ &\le \varepsilon + L(g, P) - L(\chi_E \cdot g, P) \\ &= \varepsilon\end{aligned}$$

Note that the second line follows from our construction and the definition of the integral, and the last line holds because $\inf_{A_j} g = 0$ for each $A_j$ not contained in $E$.

### Theorem 2.2.7: Dominated Convergence Theorem

Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Let $f : X \to [-\infty, \infty]$ and $f_1, f_2, \cdots : X \to [-\infty, \infty]$ be $\mathcal{S}$-measurable functions such that $\lim_{k\to\infty} f_k(x) = f(x)$ for almost every $x \in X$. If there exists an $\mathcal{S}$-measurable function $g : X \to [0, \infty]$ such that $\int g \ \mathrm{d}\mu < \infty$ and $|f_k(x)| \le g(x)$ for all $k > 0$ and almost every $x \in X$, then

$$\lim_{k\to\infty} \int f_k \ \mathrm{d}\mu = \int f \ \mathrm{d}\mu.$$

***Proof for Theorem***

Suppose $g : X \to [0, \infty]$ satisfies the hypotheses of this theorem. For any set $E \in \mathcal{S}$, we have the following estimates

$$\begin{aligned}
\left|\int f_k \ \mathrm{d}\mu - \int f \ \mathrm{d}\mu\right| &= \left|\int_{X\setminus E} f_k \ \mathrm{d}\mu - \int_{X\setminus E} f \ \mathrm{d}\mu + \int_E f_k \ \mathrm{d}\mu - \int_E f \ \mathrm{d}\mu\right| \\
&\le \int_{X\setminus E} |f_k| \ \mathrm{d}\mu + \int_{X\setminus E} |f| \ \mathrm{d}\mu + \left|\int_E (f_k - f) \ \mathrm{d}\mu\right| \\
&\le 2\int_{X\setminus E} g \ \mathrm{d}\mu + \left|\int_E (f_k - k) \ \mathrm{d}\mu\right|
\end{aligned}$$

Assume first for now that $\mu(X) < \infty$. Let $\varepsilon > 0$. From the proposition that integrals on small sets are small, there exists $\delta > 0$ such that $\int_B g \ \mathrm{d}\mu \le \frac{\varepsilon}{4}$ for every set $B \in \mathcal{S}$ such that $\mu(B) < \varepsilon$. By Egorov's theorem (since $\mu(X) < \infty$), there exists $E \in \mathcal{S}$ such that

$$\mu(X\setminus E) \le \delta$$

and $f_1, f_2, \cdots$ converges uniformly on $E$. We then have

$$\begin{aligned}
\left|\int f_k \ \mathrm{d}\mu - \int f \ \mathrm{d}\mu\right| &\le 2\int_{X\setminus E} g \ \mathrm{d}\mu + \left|\int_E (f_k - f) \ \mathrm{d}\mu\right| \\
&\le 2 \cdot \frac{\varepsilon}{4} + \mu(E) \cdot \sup_E |f_k - f| \\
&\le \frac{\varepsilon}{2} + \mu(E) \cdot \frac{2\varepsilon}{\mu(E)} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Then consider the case where $\mu(X) = \infty$. Let $\varepsilon > 0$. By proposition 3, there exists $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and $\int_{X \setminus E} \mathrm{d}\mu \leq \frac{\varepsilon}{4}$. Then we have

$$\begin{aligned}\left|\int f_k \ \mathrm{d}\mu - \int f \ \mathrm{d}\mu\right| &\leq 2\int_{X\setminus E} g \ \mathrm{d}\mu + \left|\int_E (f_k - f) \ \mathrm{d}\mu\right| \\ &\leq 2\cdot\frac{\varepsilon}{4} + \left|\int_E (f_k - k) \ \mathrm{d}\mu\right|\end{aligned}$$

By the first case as applied to the sequence $f_1|_E, f_2|_E, \cdots$. the last term on the right is less than $\frac{\varepsilon}{2}$ for all sufficiently large $k$. Thus $\lim_{k\to\infty} \int f_k \ \mathrm{d}\mu = \int f \ \mathrm{d}\mu$, completing the proof of the second case. ■

**Remark.**

- Advantages of DCT
  - We do not require the functions to be nonnegative.
  - We do not require the sequence to be increasing.
  - We do not require the measure space to be finite.
  - We do not require the sequence of functions to be uniformly bounded.

  All these hypotheses are replaced only by a requirement that the sequence of functions is pointwise bounded by a function with a finite integral.
- Bounded Convergence Theorem follows immediately from the result of Dominated Convergence Theorem. Simply take $g$ to be an appropriate constant function and use the hypothesis in the Bounded Convergence Theorem that $\mu(X) < \infty$.

### 2.2.3 Relationship Between Lebesgue and Riemann Integral

**Theorem 2.2.8: Riemann Integrable is Equivalent to Continuous Almost Everywhere**

Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is a bounded function. Then $f$ is Riemann integrable if and only if $\mu^*(\{x \in [a, b] : f \text{ is not continuous at } x\}) = 0$. Moreover, if $f$ is Riemann integrable,

$$\int_a^b f \ \mathrm{d}x = \int_{[a,b]} f \ \mathrm{d}\lambda$$

where $\lambda$ is the Lebesgue measure.