Real Analysis

Hai Le **Professor**
LaTeX by Rui Zhou
University of Michigan

Winter 2024

Contents

4 Product of Measures **2**

4.1 Products of σ-Algebras 2

4.2 Products of Measures 3

4.3 Order of Integration 5

4.4 Lebesgue Integration on $\mathbb{R}^n$ 7

Chapter 4

Product of Measures

4.1 Products of σ-Algebras

Definition 4.1.1: Rectangle

Suppose X and Y are sets. A *rectangle* in $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is a set of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$.

Definition 4.1.2

Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Then

- The product (σ-algebra) $S \otimes T$ is defined to be the smallest σ-algebra on $X \times Y$ that contains $\{A \times B : A \in \mathcal{S}, B \in \mathcal{T}\}$.
- A measurable rectangle in $\mathcal{S} \otimes \mathcal{T}$ is a set of $A \times B$ where $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

$$\mathcal{S} \times \mathcal{T} = \{(A, B) : A \in \mathcal{S}, B \in \mathcal{T}\}$$
$$\mathcal{S} \otimes \mathcal{T} \supseteq \{A \times B : A \in \mathcal{S}, B \in \mathcal{T}\}$$

Definition 4.1.3: Cross Section

Suppose X and Y are sets and $E \subseteq X \times Y$. Then for $a \in X$ and $b \in Y$, the cross sections $[E]_a$ and $[E]^b$ are defined by

$$[E]_a := \{y \in Y : (a, y) \in E\} \subseteq Y$$
$$[E]^b := \{x \in X : (x, b) \in E\} \subseteq X$$

Proposition 4.1.4: Cross Sections Preserve Measurability

Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. If $E \in \mathcal{S} \otimes \mathcal{T}$, then $[E]_a \in \mathcal{T}$ for all $a \in X$ and $[E]_b \in \mathcal{S}$ for all $b \in Y$.

Proof for Proposition.

Let $\mathcal{E}$ be the collection of subsets of $X \times Y$ for which the conclusion of the result holds. Note that $\mathcal{E}$ contains all measurable rectangles: $A \times B : A \in \mathcal{S}, B \in \mathcal{T}$.

Prove that $\mathcal{E}$ is indeed a σ-algebra:

- $\times \in \mathcal{E}$
- (Closed under complementation) If $E \in \mathcal{E}$, then $(x, y) \setminus E \in \mathcal{E}$.
- (Closed under countable union) If $E_1, E_2, \cdots \in \mathcal{S}$, then

$$[E_1 \cup E_2 \cup \cdots]_a = [E_1]_a \cup [E_2]_a \cup \cdots$$

Definition 4.1.5: Cross Section Function

Suppose X and Y are sets and $f : X \times Y \to \mathbb{R}$. For $a \in X$ and $b \in Y$, the cross section functions $[f]_a : Y \to \mathbb{R}, [f]^b : X \to \mathbb{R}$ are defined by $[f]_a(y) := f(a, y)$ for $y \in Y$, and $[f]^b(x) := f(x, b)$ for $x \in X$.

Proposition 4.1.6

f is a $\mathcal{S} \otimes \mathcal{T}$-measurable function, then $[f]_a$ is a $\mathcal{T}$-measurable function and $[f]^b$ is an $\mathcal{S}$-measurable function.

4.2 Products of Measures

Definition 4.2.1: Finite Measure

A measure μ on a measurable space $(X, \mathcal{S})$ is finite if $\mu(X) < \infty$. A measure μ is called σ-finite if the whole space can be written as a countable union of sets with finite measure.

Example.

- Lebesgue measure on $[0, 1]$ is finite.
- Lebesgue measure on $\mathbb{R}$ is σ-finite, because $\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n + 1]$.
- Counting measure on $\mathbb{R}$ is not σ-finite.

Proposition 4.2.2

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

- $x \mapsto \nu\left([E]_x\right)$ is an $\mathcal{S}$-measurable function on X.
- $y \mapsto \mu\left([E]^y\right)$ is a $\mathcal{T}$-measurable function on Y.

Definition 4.2.3

Suppose $(X, \mathcal{S}, \mu)$ is a measurable space. $g : X \to [-\infty, \infty]$. $\int g\left(x\right)\ \mathrm{d}\mu\left(x\right)$ means $\int g\ \mathrm{d}\mu$, where $\mathrm{d}\mu\left(x\right)$ indicates that variables other than x should be treated as constants.

Definition 4.2.4: Iterated Integrals

Definition (Iterated Integrals) Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. $f : X \times Y \to \mathbb{R}$.

$$\int_X \int_Y f\left(x, y\right)\ \mathrm{d}\nu\left(y\right)\ \mathrm{d}\mu\left(x\right) = \int_X \left(\int_Y f\left(x, y\right)\ \mathrm{d}\nu\left(y\right)\right)\ \mathrm{d}\mu\left(x\right).$$

Example.

> If λ is the Lebesgue measure on $\mathbb{R}$, then
>
> $$\int_{[0,4]} \int_{[0,4]} \left(x^2 + y\right)\ \mathrm{d}\lambda\left(y\right)\ \mathrm{d}\lambda\left(x\right) = \int_{[0,4]} \left(4x^2 + 8\right)\ \mathrm{d}\lambda\left(x\right) = \frac{352}{3}.$$

Definition 4.2.5

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are measure spaces. For $E \in \mathcal{S} \otimes \mathcal{T}$, define $\left(\mu \times \nu\right)\left(E\right)$ by

$$\left(\mu \times \nu\right)\left(E\right) := \int_X \int_Y \chi_E\left(x, y\right)\ \mathrm{d}\nu\left(y\right)\ \mathrm{d}\lambda\left(x\right).$$

Proposition 4.2.6

$\mu \times \nu$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$.

Proof for Proposition.

> It is trivial to see that $\left(\mu \times \nu\right)\left(\right) = 0$.

$(\mu \times \nu)$ is countably additive: Suppose E_k's are disjoint sets in $\mathcal{S} \otimes \mathcal{T}$,

$$\begin{aligned}(\mu \times \nu)\left(\bigcup_{k=1}^{\infty} E_k\right) &= \int_X \int_Y \chi_{\bigcup_{k=1}^{\infty} E_k}(x, y)\ \mathrm{d}\nu(y)\ \mathrm{d}\lambda(x) \\ &= \int_X \nu\left(\left[\bigcup_{k=1}^{\infty} E_k\right]_x\right)\ \mathrm{d}\mu(x) \\ &= \int_X \nu\left(\bigcup_{k=1}^{\infty} [E_k]_x\right)\ \mathrm{d}\mu(x) \\ &= \int_X \sum_{k=1}^{\infty} v([E_k]_x)\ \mathrm{d}\mu(x) \\ &= \sum_{k=1}^{\infty} \int_X v([E_k]_x)\ \mathrm{d}\mu(x) \\ &= \sum_{k=1}^{\infty} (\mu \times \nu)(E_k)\end{aligned}$$

Note that you may change the order of infinite series and integral by monotone convergence theorem.

4.3 Order of Integration

Theorem 4.3.1: Tonellis's Theorem

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f : X \times Y \to [0, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable. Then

- $x \mapsto \int_Y f(x, y)\ \mathrm{d}\nu(y)$ is an $\mathcal{S}$-measurable function on X.
- $y \mapsto \int_X f(x, y)\ \mathrm{d}\mu(x)$ is a $\mathcal{T}$-measurable function on Y.
- $\int_{X \times Y} f(x, y)\ \mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x, y)(y)\right)\ \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y)\ \mathrm{d}\mu(x)\right)(y)$.

Proof for Theorem

Essentially, we only need to check if the theorem is true for $f = \chi_{A \times B}(x, y)$.

$$\begin{aligned}\int_{X \times Y} \chi_{A \times B}\ \mathrm{d}\mu \times \nu &= \int_X \left(\int_Y \chi_{A \times B}(y)\right)\ \mathrm{d}\mu(x) \\ &= \int_X \nu(B)\ \mathrm{d}\mu(x) \\ &= \nu(B)\, \mu(A) \\ &= \mu \times \nu(A \times B)\end{aligned}$$

note that the equality in the last line is simply by definition of $\mu \times \nu$.

Example.

- Without σ-finite:
  - $([0,1], \mathcal{B}.\lambda)$: Lebesgue measure space on $[0,1]$.
  - $([0,1], \mathcal{B}, \mu)$: Counting measure space on $[0,1]$.
  - Let D be the diagonal of $[0,1] \times [0,1]$, $D = \{(x,x) : x \in [0,1]\}$.

$$\int_{[0,1]} \int_{[0,1]} \chi_D \ \mathrm{d}\lambda \ \mathrm{d}\mu = \int_{[0,1]} 0 \ \mathrm{d}\mu = 0$$
$$\int_{[0,1]} \int_{[0,1]} \chi_D \ \mathrm{d}\mu \ \mathrm{d}\lambda = \int_{[0,1]} 1 \ \mathrm{d}\lambda = 1$$

- Without non-negativity:

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2} \ \mathrm{d}x \ \mathrm{d}y \neq \int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2} \ \mathrm{d}y \ \mathrm{d}x$$

Theorem 4.3.2: Fubini's Theorem

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f : X \times Y \to [-\infty, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable, and $\int_{X \times Y} |f| \ \mathrm{d}\mu \times \nu < \infty$. Then

- $\int_Y |f(x,y)|(y) < \infty$ for almost every $x \in X$, and $\int_X |f(x,y)| \ \mathrm{d}\mu(y) < \infty$ for almost every $y \in Y$.
- $x \mapsto \int_Y f(x,y) \ \mathrm{d}\nu(y)$ is an $\mathcal{S}$-measurable function on X. $y \mapsto \int_X f(x,y) \ \mathrm{d}\mu(x)$ is a $\mathcal{T}$-measurable function on Y.
- $\int_{X \times Y} f(x,y) \ \mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x,y)(y)\right) \ \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x,y) \ \mathrm{d}\mu(x)\right)(y)$.

Proof for Theorem

Apply Tonellis's theorem to f^+, f^- and $|f| < \infty$ makes sure that both $\int f^+$ and $\int f^-$ are finite.

Definition 4.3.3: Under-Graph Region

Definition () Suppose X is a set and $f : X \to [0, \infty]$ is a function. The region under the graph f, denoted as U_f, is defined by

$$U_f = \{(x,t) \in (X, (0, \infty)) : 0 < t < f(x)\}$$

Proposition 4.3.4

Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space, and $f : X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Consider $((0, \infty), \mathcal{B}, \lambda)$ (where λ is the Lebesgue measure). Then

$$U_f \in \mathcal{S} \times \mathcal{B}$$

and

$$\begin{aligned} \mu \times \nu (U_f) &= \int_X f \, \mathrm{d}\mu \\ &= \int_{(0,\infty)} \mu (\{x \in X : 0 < t < f(x)\}) \, \mathrm{d}\lambda (t) \end{aligned}$$

Proof for Proposition.

Apply Tonellis' theorem to $\chi_{U_f}(x, t)$ to finish the proof.

4.4 Lebesgue Integration on $\mathbb{R}^n$

Consider the n-dimensional space

$$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_1, x_2, \cdots, x_n \in \mathbb{R}\}$$

Pick *any* norm $\|\cdot\|_2$ on $\mathbb{R}^n$. We can define open balls:

$$B(x, \delta) := \{y \in \mathbb{R}^n : \|x - y\|_2 < \delta\}$$

Definition 4.4.1: Borel σ-algebra $\mathcal{B}_n$ of $\mathbb{R}^n$

The Borel σ-algebra $\mathcal{B}_n$ of $\mathbb{R}^n$ is the smallest σ-algebra that contains all open balls.

Proposition 4.4.2

$$\mathcal{B}_m \otimes \mathcal{B}_n = \mathcal{B}_{m+n}$$

Relate this result to $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

Definition 4.4.3: Lebesgue measure on $\mathbb{R}^n$

The Lebesgue measure on $\mathbb{R}^n$, λ_n is defined inductively:

$$\lambda_n := \lambda_{n-1} \times \lambda_1$$

where λ_1 is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}_1)$.

$$\lambda_{m+n} = \lambda_m \times \lambda_n.$$