Real Analysis

Hai Le **Professor**
LaTeX by Rui Zhou
University of Michigan

Winter 2024

Chapter 5

Lp Spaces

Definition 5.0.1: $\|f\|_{1p}$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $0 < p < \infty$, and $f : X \to$ is $\mathcal{S}$-measurable. Then the p-norm of f, denoted $\|f\|_{1p}$, is defined by

$$\|f\|_{1p} = \left(\int |f|^p \; \mathrm{d}\mu \right)^{1/p}.$$

Also when $p = \infty$, $\|f\|_{1\infty}$ is called the *essential supremum* of f, and is defined by

$$\|f\|_{1\infty} := \inf \{t > 0 : \mu (\{x \in X : |f(x)| > t\}) = 0\}.$$

Definition 5.0.2: $L^p(\mu)$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 < p \leq \infty$. The Lebesgue space $L^p(\mu)$ is denoted to be the set of all $\mathcal{S}$-measurable functions $f : X \to$ such that

$$\|f\|_{1p} < \infty.$$

$L^p(\mu)$ is a vector space. However, it is NOT a normed vector space. There exists some $f, g \in L^p(\mu)$ such that $f \neq g$ but $\|f - g\|_{1p} = 0$.

We construct *equivalence classese*:

$$f \sim g \iff f = g \text{ a.e.}$$

and identify each function with its own equivalence class.

Definition 5.0.3: Dual Exponent

Suppose $1 \leq p \leq \infty$. The *dual exponent* of p is denoted by $p' \in [1, \infty]$ that satisfies

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Example.

$2' = 2$, $1' = \infty$, $\infty' = 1$.

Proposition 5.0.4

- (Holder's Inequality) $1 \leq p \leq \infty$, $f, h : X \to$, then

$$\|fh\|_{11} \leq \|f\|_{1p}\|h\|_{1p'}.$$

- $0 < p < q < \infty$, $\mu(X) < \infty$, then

$$\|f\|_{1p} \leq \mu(X)^{(q-p)/(pq)} \|f\|_{1q}.$$

for all $f \in L^q(\mu)$. Furthermore, $L^q(\mu) \leq L^p(\mu)$.

- $1 \leq p < \infty$, $f \in :^p(\mu)$. Then

$$\|f\|_{1p} = \sup\left\{\left|\int fh \, \mathrm{d}\mu\right| : h \in L^{p'}(\mu) \text{ and } \|h\|_{1p} \leq 1\right\}.$$

- (Minowsky's Inequality: $\|\cdot\|_{1p}$ is a norm) $1 \leq p \leq \infty$,

$$\|f+g\|_{1p} \leq \|f\|_{1p} + \|g\|_{1p}$$

Now that we know L^p is a normed vector space. We want to show further that L^p is a Banach space.

Proposition 5.0.5

$1 \leq p \leq \infty$, $\{f_j\} \in L^p(\mu)$ such that for all $\varepsilon > 0$, there exists $n \in$ such that

$$\|f_j - f_k\|_1 < \varepsilon, \forall j, k \geq n.$$

There exists $f \in L^p$ such that

$$\lim_{k \to \infty} \|f_k - f\|_{1p} = 0.$$

Proof for Proposition.

To prove a Cauchy sequence is convergent, it suffice to show the convergence of any subsequence.

To extract a subsequence with the following properties: Suppose $f_0 = 0$,

$$\sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_1 < \infty$$

This step is valid because the sequence of Cauchy.

Define $g_1, g_2, \cdots, g_m$ such that

$$g_m(x) = \sum_{k=1}^{m} |f_k(x) - f_{k-1}(x)|$$

and

$$g(x) = \sum_{k=1}^{\infty} |f_k(x) - f_{k-1}(x)|$$

We have by Triangle Inequality,

$$\|g_m\|_{1p} \leq \sum_{k=1}^{m} \|f_k(x) - f_{k-1}(x)\|_{1p} < \infty$$

Moreover, we have pointwise convergence:

$$g_m(x) \to g(x), \forall x \in X$$

By DCT, we have

$$\|g\|_{1p} = \lim_{m\to\infty} \|g_m\|_{1p} < \infty$$

$$\Longrightarrow g < \infty \text{ a.s}$$

Therefore,

$$\sum_{k=1}^{m} (f_k(x) - f_{k-1}(x)) = \lim_{m\to\infty} f_m(x) = f(x)$$

Moreover, $|f| \leq g$ and $g \in L^p$, this implies that $f \in L^p$.

Finally, $\|f_k - f\|_{1p} \leq \liminf_{j\to\infty} \|f_k - f_j\|_{1p} \leq \varepsilon$, for k large enough. This implies that

$$\lim_{k\to\infty} \|f_k - f\|_{1p} = 0.$$