Real Analysis

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Chapter 5

Banach Spaces

5.1 Metric Spaces

Definition 5.1.1: Metric

A metric on a nonempty set V is a function $d : V \times V \to [0, \infty)$ such that

- $d(f, f) = 0$ for all $f \in V$.
- If $f, g \in V$ and $d(f, g) = 0$, then $f = g$.
- $d(f, g) = f(g, f)$.
- (Triangle Inequality) $d(f, h) \leq d(f, g) + d(g, h)$, for all $f, g, h \in V$.

(V, d) is called a metric space.

Example.

- V is any nonempty set, $d(f, g) = \begin{cases} 1 \text{ if } f \neq g \\ 0 \text{ if } f = g \end{cases}$.
- $V = \mathbb{R}$, $d(f, g) = |f - g|$.
- l^∞-metric: $V = \mathbb{R}^n$,

$$d((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) = \max\{|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|\}.$$

- $V = C([0, 1])$, which stands for the collection of all continuous functions defined on $[0, 1]$.

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [0, 1]\}$$

- $l^1 = \{(a_1, a_2, \cdots) : \sum_{k=1}^{\infty} |a_k| < \infty\}$.

$$d\left(\left(a_1, a_2, \cdots\right), \left(b_1, b_2, \cdots\right)\right) = \sum_{k=1}^{\infty} |a_k - b_k| < \infty$$

Definition 5.1.2: Open Ball

Let (V, d) be a metric space, $f \in V$, $r > 0$.

- The open ball center center f radius r is denoted $B(f, r)$ and defined by
$$B(f, r) = \{g \in V : d(f, g) < r\}.$$
- The closed ball center center f radius r is denoted $\bar{B}(f, r)$ and defined by
$$\bar{B}(f, r) = \{g \in V : d(f, g) \leq r\}.$$

Definition 5.1.3: Open Set

A subset $G \subseteq V$ is called open if for every $f \in G$, there exits $r > 0$ such that $B(f, r) \subseteq G$.

Definition 5.1.4: Closed Set

A subset $F \subseteq V$ is closed if and only if the complement of F in V is open.

Definition 5.1.5: Closure

Let $E \subseteq V$. The closure of E, denoted as $\bar{E}$, is defined by

$$\bar{E} := \{g \in V : B(g, \varepsilon) \cap E \neq 0, \forall \varepsilon > 0\}.$$

$\{f_k\} \subseteq V$, $f \in V$. *We write* $\lim_{k\to\infty} f_k = k$ *to mean* $\lim_{k\to\infty} d(f, f_k) = 0$.

Proposition 5.1.6

Let $\bar{E}$ be the closure of a set E. Then

- $\bar{E} = \{g \in V : \{f_k\} \subseteq E \text{ s.t. } \lim_{k\to\infty} f_k = g\}$.
- $\bar{E}$ is the intersection of all closed sets containing E.
- $\bar{E}$ is closed.
- E is closed if and only if $E = \bar{E}$.
- E is closed if and only if E contains the limit of every convergent sequence of elements of E.

Definition 5.1.7: Continuity

Suppose (V, d_v) and (W, d_w) are metric spaces, and $T : V \to W$ is a function. For $f \in V$, T is continuous at f if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_w\left(T\left(f\right), T\left(g\right)\right) < \varepsilon$$

given $d_v\left(f, g\right) < \delta$ and $g \in V$.

Proposition 5.1.8

All of the followings are equivalent:

- T is continuous on V.
- If $\lim_{k\to\infty} f_k = f$, then $\lim_{k\to\infty} T\left(f_k\right) = T\left(f\right)$.
- $T^{-1}\left(G\right)$ is open in V if G is open in W.
- $T^{-1}\left(F\right)$ is closed in V if F is closed in W.

Definition 5.1.9: Cauchy Sequence

A sequence $f_1, f_2, \cdots$ is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$d\left(f_j, f_k\right) \leq \varepsilon$$

for all $k, j \geq N$.

Proposition 5.1.10

Every convergence sequence in a metric space is a Cauchy sequence.

Definition 5.1.11: Complete Metric Space

A metric space V is called *complete* if every Cauchy sequence in V converges to some element in V.

Example.

> $V =$, $d\left(x, y\right) = |x - y|$. Consider
>
> $$x_k = \frac{1}{10^{1!}} + \cdots + \frac{1}{10^{k!}}.$$
>
> $\{x_k\}$ is Cauchy, but does not converge to a rational number.

Proposition 5.1.12

- A complete subset of a metric space is closed.
- A closed subset of a metric space is complete.

5.2 Vector Spaces

Definition 5.2.1: S-Measurable Function in $\mathbb{C}$

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f : X \to \mathbb{C}$ is called $\mathcal{S}$-measurable if and only if $\text{Re } f : X \to \mathbb{R}$ and $\text{Im } f : X \to \mathbb{R}$ are $\mathcal{S}$-measurable functions.

All measurability results are true for complex-valued functions.

Definition 5.2.2: $\int f \, \mathbf{d}\mu$

Let $f : X \to \mathbb{C}$ be such that $\int |f| < \infty$. Then $\int f \, \mathrm{d}\mu$ is defined by

$$\int f \, \mathrm{d}\mu := \int \text{Re } f \, \mathrm{d}\mu + i \int \text{Im } f \, \mathrm{d}\mu.$$

Definition 5.2.3: Complex Conjugate

Let $\bar{z}$ be the *complex conjugate* of a complex number z. If $z = a + bi$, then $\bar{z} = a - bi$.

$$\overline{\int f \ d\mu} = \int \bar{f} \ d\mu \ \textit{if} \ \int |f| \ \ d\mu < \infty.$$

From now on, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Definition 5.2.4: Vector Space

A vector space (over $\mathbb{F}$) is a set V with an addition $(+)$ on V and a scalar multiplication $(\alpha f,\, \alpha \in \mathbb{F})$ on V that satisfies

- Commutativity
- Associativity
- Additive identity
- Additive inverse
- Multiplicative identity
- Distributive property

Definition 5.2.5: $\mathbb{F}^X$

$$\mathbb{F}^X = \{\text{All functions from } X \text{ to } \mathbb{F}\}.$$

Functional analysis could be thought of as linear algebra on infinite dimensional spaces.

5.2.1 Normed Vector Spaces

Definition 5.2.6: Norm

A norm on a vector space V (over $\mathbb{F}$) is a function

$$\|\cdot\|: V \to [0, \infty)$$

such that

- (Positive Definiteness) $\|f\|_1 = 0 \iff f = 0$,
- (Homogeneity) $\|\alpha f\|_1 = |a| \|f\|_1$ for all $\alpha \in \mathbb{F}$ and $f \in V$, and
- (Triangle Inequality) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$ for all $f, g \in V$.

Most of the time the first two properties are easy to check but the hard things come with the proof of Triangle Inequality.

Example.

- $(\mathbb{R}, |\cdot|)$; $(\mathbb{C}, |\cdot|)$.
- $\mathbb{F}^n = \{(a_1, a_2, \cdots, a_n) : a_1, a_2, \cdots, a_n \in \mathbb{F}\}$, with

$$\|(a_1, a_2, \cdots, a_n)\|_{11} = \sum_{k=1}^{n} |a_k|$$

$$\|(a_1, a_2, \cdots, a_n)\|_{12} = \sqrt{\sum_{k=1}^{n} |a_k|^2}$$

$$\|(a_1, a_2, \cdots, a_n)\|_{1p} = \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p}$$

$$\|(a_1, a_2, \cdots, a_n)\|_{1\infty} = \max\{|a_1|, |a_2|, \cdots, |a_n|\}$$

- $\ell^1 = \{(a_1, a_2, \cdots) : \sum_{n=1}^{\infty} |a_n| < \infty\}$, with

$$\|(a_1, a_2, \cdots)\|_1 = \sum_{n=1}^{\infty} a_n.$$

- X is any set. Let $b(X)$ be the set of all bounded functions. $b(X) = \{\text{Bounded functions} : X \to \mathbb{F}\}$.

$$\|f\|_1 = \sup\{|f(x)| : x \in X\}.$$

- $C([0,1]) = \{\text{Continuou functions} : [0,1) \to \mathbb{F}\}$, with

$$\|f\|_1 = \int_0^1 |f|.$$

A norm will induce a metric on the same vector space. One straightforward example would be

$$d(f,g) = \|f-g\|_1$$

Therefore, any normed vector space is a metric space. However the converse is false. With this in mind we may apply all stuffs in metric space to any normed vector space.

We have the meaning of limit:

$$\lim_{k\to\infty} f_k = f \iff \lim_{k\to\infty} \|f_k - f\|_1 = 0.$$

We also have the notion of Cauchy sequence.

Definition 5.2.7: Banach Space

A *Banach space* is a normed complete vector space.

Example.

- $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^n$ are Banach spaces.
- $C([0,1]) = \{\text{Continuou functions} : [0,1) \to \mathbb{F}\}$ with $\|f\|_1 = \max_{[0,1]} |f|$ is a Banach space.
- $\left(\ell^1, \|\cdot\|_{11}\right)$ is a Banach space.
- $C([0,1]) = \{\text{Continuou functions} : [0,1) \to \mathbb{F}\}$ with $\|f\|_1 = \int_0^1 |f|$ is *not* a Banach space.
- $\left(\ell^1, \|\cdot\|_{1\infty} = \sup_{k\in\mathbb{N}} |a_k|\right)$ is *not* a Banach space.

Definition 5.2.8: Series

Suppose $g_1, g_2, \cdots$ is a sequence in a normed vector space V. Then

$$\sum_{k=1}^{\infty} g_k \text{ exists} \iff \lim_{n\to\infty} \sum_{k=1}^{n} \text{ exists.}$$

When $\lim_{n\to\infty} \sum_{k=1}^{n}$ exists, the series $\sum_{k=1}^{\infty} g_k$ is said to converge.

Proposition 5.2.9

V is a Banach space if and only $\sum_{k=1}^{\infty} g_k$ converges to some element in V for any sequence satisfying $\sum_{k=1}^{\infty} \|g_k\|_1 < \infty$.

Proof for Proposition.

- $\Longrightarrow$: Assume $g_1, g_2, \cdots$ satisfies $\sum_{k=1}^{\infty} \|g_k\|_1 < \infty$. Let $f_j = \sum_{n=1}^{j} g_n$. Fix $\varepsilon > 0$.

$$\|f_j - f_k\|_1 = \| \sum_{n=k+1}^{j} g_n \|_1 \leq \sum_{n=k+1}^{j} \|g_n\|_1 < \varepsilon$$

  which is true for k, j sufficiently large.

- $\Longleftarrow$: Suppose $f_1, f_2, \cdots$ is Cauchy in V. We can extract a subsequence of $\{f_j\}$ (without relabeling) such that

$$\sum_{k=1}^{\infty} \|f_{k+1} - f_k\|_1 < \infty.$$

  This implies that $\sum_{k=1}^{\infty} (f_{k+1} f_k)$ converges to an element in V. Moreover,

$$\lim_{N\to\infty} \sum_{k=1}^{N} (f_{k+1} - f_k) = \lim_{N\to\infty} (f_{N+1} - f_1),$$

  which implies that $\lim_{N\to\infty} f_{N+1}$ exists and $\lim_{N\to\infty} f_{N+1}$ is also an element in V. The last step is that, since the original sequence is Cauchy, and we have proved that one of its subsequence is convergent, so is the original Cauchy sequence.

5.2.2 Bounded Linear Maps

Definition 5.2.10: Linear Map

Suppose V and W are vector spaces. A function $T : V \to W$ is called a linear map or linear if

- $T(f + g) = T(f) + T(g)$ for $f, g \in V$.
- $T(\alpha f) = \alpha T(f)$ for $\alpha \in \mathbb{F}$ and $f \in V$.

For simplicity we may write Tf and $T(f)$ interchangeably with $Tf = T(f)$. $\|\cdot\|_1$ is used for both norms of V and W, whose meaning depends on the context.

Basically, a linear map is a mapping that preserves the structure of a linear space.

Definition 5.2.11: Norm of a Linear Map

Let V and W be normed vector spaces and $T : V \to W$ be a linear map. The norm of T, denoted $\|T\|_1$, is defined by

$$\begin{aligned}\|T\|_1 &= \sup\{\|T(f)\|_1 : f \in V \text{ and } \|f\|_1 \le 1\} \\ &= \sup\{\|T(f)\|_1 : f \in V \text{ and } \|f\|_1 = 1\} \\ &= \sup_{f \neq 0}\left\{\frac{\|T(f)\|_1}{\|f\|_1}\right\}\end{aligned}$$

T is called bounded if $\|T\|_1 < \infty$. The set of bounded linear maps from V to W is defined by $\mathcal{B}(V, W)$.

Example.

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. There exists a matrix $A \in \mathbb{R}^{m\times n}$ such that

  $$T(v) = Av,$$

  and $\mathcal{B}(\mathbb{R}^n, \mathbb{R}^m) \simeq \mathbb{R}^{m\times n}$. Let $\|A\|_1$ be the largest singular value of A, which is equal to the square root of the largest eigenvalue of $A'A$.

- $C([0, 3))$ with $\|f\|_1 = \max_{[0,3]}|f|$. Consider the linear map $T : C([0, 3]) \to C([0, 3])$ so that $Tf : [0, 3] \to \mathbb{R}$ such that

  $$(Tf)(x) = x^2 f(x), \forall x \in [0, 3).$$

  Prove that $\|T\|_1 = 9$.

- $V = \{(a_1, a_2, \cdots) : a_k = 0 \text{ for all but finitely many } k \in \mathbb{N}\}$, with $\|(a_1, a_2, \cdots)\|_{1\infty} = \max|a_k|$. Consider the linear map $T : V \to V$ such that

  $$T(a_1, a_2, a_3, \cdots) = (a_1, 2a_2, 3a_3, \cdots).$$

  Show that $\|T\|_1 = \infty$.

Proposition 5.2.12

Assuming V is a normed vector space and W is a Banach space. $\mathcal{B}(V, W)$ is a Banach space with $\|T\|_1$.

Proof for Proposition.

Consider $T_1, T_2, \cdots \in \mathcal{B}(V, W)$ to be Cauchy sequence. We want to show that, there is $T \in \mathcal{B}(V, W)$ such that $\lim_{k\to\infty}\|T_k - T\|_1 = 0$.

Pick any $f \in V$. Consider the sequence $T_1 f, T_2, \cdots$. We have

$$\|T_j f - T_k f\|_1 \leq \|T_j - T_k\|_1 \|f\|_1,$$

which implies that $T_1 f, T_2 f, \cdots$ is a Cauchy sequence in W. Then, there exists $Tf \in W$ such that

$$\lim_{k \to \infty} T_k f = Tf \in W.$$

Define $T : V \to W$ to be

$$T(f) = \lim_{k \to \infty} T_k f.$$

T is first of all a linear map. Moreover, T is bounded because

$$\begin{aligned} \|Tf\|_1 &\leq \sup_k \{\|T_k f\|_1\} \\ &\leq \left(\sup_k \|T_k\|_1\right) \|f\|_1 < \infty \end{aligned}$$

The last thing to show is that $\lim_{k \to \infty} \|T_k - T\|_1 = 0$. Fix $k > 0$. For any $\varepsilon > 0$, the following is true for k large enough:

$$\begin{aligned} \|(T_k - T) f\|_1 &= \lim_{j \to \infty} \|(T_k - T_j) f\|_1 \\ &\leq \lim_{j \to \infty} \|(T_k - T_j)\|_1 \|f\|_1 \\ &\leq \varepsilon \|f\|_1 \end{aligned}$$

which implies that

$$\|T_k - T\|_1 \leq \varepsilon \implies \lim_{k \to \infty} T_k = T.$$

Proposition 5.2.13

Suppose $T : V \to W$ is a linear map. T is bounded if and only if T is continuous.

Proof for Proposition.

- $\Longleftarrow$: Assume T is continuous and not bounded. There exists a sequence $f_1, f_2, \cdots \in V$ such that $\|f_k\|_1 \leq 1$ for all $k \in \mathbb{N}$ but $\|T_k f\|_1 \to \infty$ as $k \to \infty$. Therefore,

$$\lim_{k \to \infty} \frac{f_k}{\|Tf_k\|_1} = 0,$$

  but

$$T\left(\frac{f_k}{\|Tf_k\|_1}\right) = \frac{T(f_k)}{\|Tf_k\|_1}.$$

- $\Longrightarrow$: Suppose $\lim_{k \to \infty} f_k = f$. We have that

$$\|Tf_k - Tf\|_1 \leq \|T\|_1 \|f_k - f\|_1$$

For the right-hand side, $\lim_{k\to\infty} \|T\|_1 \|f_k - f\|_1 = 0$.

5.2.3 Bounded Linear Functionals

Definition 5.2.14: Linear Functional

A *linear functional* is a linear map from a normed vector space V to a scalar vector space ($\mathbb{R}$ or $\mathbb{F}$).

Example.

- $V = \{(a_1, a_2, \cdots) : a_k = 0 \text{ but for finitely many } k \geq 0\}$. Consider

$$\phi(a_1, a_2, \cdots) = \sum_{k=1}^{\infty} a_k.$$

- If V has the norm $\|(a_1, a_2, \cdots)\|_1 = \sum_{k=1}^{\infty} |a_k|$, ϕ is bounded.
- If V has the norm $\|(a_1, a_2, \cdots)\|_1 = \max_{k\geq 0} |a_k|$, ϕ is not bounded.

Definition 5.2.15: Null Space

Suppose $T : V \to W$ is a linear map. The null space, or kernel, of T, denoted null T, is defined by

$$\text{null } T = \{f \in V : Tf = 0\}.$$

null T is a subspace of V. If T is continuous, then null T is a closed subspace.

$$\textit{null } T = T^{-1}(\{0\}).$$

Proposition 5.2.16

Suppose $\phi : V \to \mathbb{F}$ is a linear functional that is not identically 0. Then the followings are equivalent:

- ϕ is bounded.
- ϕ is continuous.
- null ϕ is a closed subspace of V.
- $\overline{\text{null } \phi} \neq V$.

Proof for Proposition.

- $3 \implies 1$: Suppose null ϕ is a closed and ϕ is unbounded. We hope to reach a contradiction.

  There exits a sequence $f_1, f_2, \cdots \in V$ such that $\|f_k\|_1 = 1$ and $|\phi(f_k)| \geq k$ for each $k \geq 1$. We have

  $$\frac{f_1}{\phi(f_1)} - \frac{f_k}{\phi(f_k)} \in \text{null } \phi,$$

  which is only possible if $\phi(\cdot)$ is a number.

  $$\lim_{k\to\infty}\left(\frac{f_1}{\phi(f_1)} - \frac{f_k}{\phi(f_k)}\right) = \frac{f_1}{\phi(f_1)}.$$

  But, $\phi\left(\frac{f_1}{\phi(f_1)}\right) = 1$, which is obviously a contradiction.

Infinite dimensional normed vector space always has a discontinuous or unbounded linear functional.

Definition 5.2.17

Suppose $\{e_k\}_{k\in\Gamma}$ is a family of elements in V.

- $\{e_k\}$ is called *linear independent* if there does NOT exist a finite nonempty subset Ω of Γ and a family $\{\alpha_j\}_{j\in\Omega} \in \mathbb{F}\setminus\{0\}$ such that $\sum_{j\in\Omega}\alpha_j e_j = 0$.
- span $\{e_k\} = \left\{\sum_{j\in\Omega}\alpha_j e_j : \Omega \subseteq \Gamma, \Omega \text{ is finite}, \alpha_j \in \mathbb{F}\right\}$.
- V is finite dimensional if and only if there exists a finite set Γ and a family $\{e_k\}_{k\in\Gamma}$ such that $V = \text{span } \{e_k\}$. V is *infinite dimensional* if V is NOT finite dimensional.
- A family $\{e_k\}$ in V is called a *basis* if it is linearly independent and span $\{e_k\} = V$.

By Axiom of choice (or Zoin's Lemma), there exists a basis for any vector space, including infinite dimensional spaces.

Proposition 5.2.18

If V is infinite dimensional, then there exits $\phi : V \to \mathbb{F}$ such that ϕ is unbounded.

Proof for Proposition.

Suppose $\{e_k\}_{k\in\Gamma}$ is a basis for V and Γ is infinite. We can assume by relabeling that $\mathbb{N} \subseteq \Gamma$, because each infinite set has at least an countably infinite one.

Define $\phi : V \to \mathbb{F}$ by

$$\phi(e_j) = \begin{cases} j\|e_j\|_1 & \text{if } j \in \mathbb{N} \\ 0 & \text{if } j \in \Gamma\setminus \end{cases}$$

By definition,

$$\phi\left(\sum_{j\in\Omega} a_j e_j\right) = \sum_{j\in\Omega} a_j \phi(e_j).$$

It is trivial to show that $\|\phi\|_1 = \infty$. ($\|\phi\|_1 \geq \frac{|\phi(e_j)|}{\|e_j\|_1} = j$)

5.2.4 Hahn-Banach Theorem

Lemma 5.2.19: Extension Lemma

Suppose V is a *real* normed vector space; U is a subspace of V; and $\phi : U \to \mathbb{R}$ is a bounded linear functional. Suppose $h \in V\backslash U$. There exists a bounded linear functional $\varphi : U + \mathbb{R}h \to \mathbb{R}$ such that

$$\begin{cases} \varphi_U = \phi \\ \|\varphi\|_1 = \|\phi\|_1 \end{cases}$$

Theorem 5.2.20: Hahn-Banach Theorem

Suppose V is a *real* normed vector space; U is a subspace of V; and $\phi : U \to \mathbb{R}$ is a bounded linear functional. Suppose $h \in V\backslash U$. There exists a bounded linear functional $\varphi : V \to \mathbb{R}$ such that

$$\begin{cases} \varphi_U = \phi \\ \|\varphi\|_1 = \|\phi\|_1 \end{cases}$$

5.2.5 Baire's Category Theorem

Definition 5.2.21: Interior

Suppose U is a subset of a metric space V. The interior of U, denoted int U, is the set of $f \in U$ such that some open ball of V centered at f with positive radius is contained in U.

int U is the largest open set *contained in U. Note that the closure of U is the smallest closed set that contains U.*

Definition 5.2.22: Dense Subset

A subset U is *dense* in V if and only if $\bar{U} = V$.

Example.

and $\mathbb{R}\backslash$ are both dense in $\mathbb{R}$.

Proposition 5.2.23: Properties of Dense Subset

- U is dense in V if and only if every open subset of V contains at least one element of U.
- U has empty interior (i.e., int $U =$) if and only if $V \backslash U$ is dense in V.

Theorem 5.2.24: Baire's Category Theorem

- A complete metric space is NOT the *countable* union of closed subsets with empty interior.
- The countable intersection of dense open subsets of a complete metric space is *non-empty*.

The intuition of the theorem is stating that, the countable union of small sets remains small, and the countable intersection of big sets remains big.

5.2.6 Theorems

Theorem 5.2.25: Open Mapping Theorem

Suppose V and W are Banach spaces. Consider $T : V \to W$, which is a bounded (or equivalently, continuous) linear map and subjective (or say, onto). Then $T(G)$ is open in W if G is open in V.

Proof for Theorem

- Start with the simplest case. Let B be the unit ball in V. That is,

$$B = \mathcal{B}(0, 1) = \{f \in V : \|f\|_1 < 1\}.$$

It suffices to show that $T(B)$ contains an open ball centered at 0, which is equivalent to prove that $0 \in \text{int } T(B)$.

Indeed, by linearity of T,

$$T(\mathcal{B}(f, a)) = Tf + aT(B).$$

for any $f \in V$ and $a > 0$.

Let $G \subseteq V$ be open, and $f \in G$. By definition of a set being open, there exists an open ball $\mathcal{B}(f, a) \subseteq G$. If $0 \in \text{int } T(B)$,

$$Tf = Tf + 0 \in \text{int } T(\mathcal{B}(f, a)) \subseteq \text{int } T(G).$$

By property of interior, G must be open.

- The surjectivity and linearity of T implies that

$$W = \bigcup_{k=1}^{\infty} T(kB) = \bigcup_{k=1}^{\infty} kT(B).$$

  Because W is Banach, $\bar{W} = W$. Hence,

$$W = \bar{W} = \bigcup_{k=1}^{\infty} \overline{kT(B)}.$$

  By Baire's Theorem, there exists k such that

$$\text{int } \overline{kT(B)} \neq .$$

  Because of linearity of T, it follows that $\text{int } \overline{T(B)} \neq$.

  Then there exists $f \in B$ such that $Tf \in \text{int } \overline{T(B)}$. Hence,

$$0_W \in \overline{T(B-f)} \subseteq \text{int } \overline{T(2B)} = \text{int } \overline{2T(B)}.$$

  It is implied that there exists $r > 0$ such that

$$\bar{\mathcal{B}}(0_W, 2r) \subseteq \text{int } \overline{2T(B)}.$$

  which immediately implies that

$$\bar{\mathcal{B}}(0_W, r) \subseteq \text{int } \overline{T(B)}.$$

- By definition of closure, take $h \in W$ and $\|h\|_1 \leq r$. There exists $f \in B$ such that

$$\|h - Tf\|_1 < \varepsilon.$$

  Equivalently speaking, for any $h \in W$, there exists $f \in \frac{\|h\|_1}{r} B$ such that

$$\|h - Tf\|_1 < \varepsilon.$$

  Finally, pick any $g \in W$ and $\|g\|_1 < 1$.

  – Use $(*)$ with $h = g$ and $\varepsilon = \frac{1}{2}$. There exists $f_1 \in \frac{1}{r}B$ such that $\|g - Tf_1\|_1 < \frac{1}{2}$.
  – Use $(*)$ with $h = g - Tf_1$ and $\varepsilon = \frac{1}{4}$. There exists $f_2 \in \frac{1}{2r}B$ such that $\|g - Tf_1 - Tf_2\|_1 < \frac{1}{4}$.
  – We continue this process inductively and obtain a sequence $f_1, f_2, \cdots$ such that

$$\sum_{k=1}^{\infty} \|f_k\|_1 \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}r} = \frac{2}{r} < \infty.$$

Eventually,

$$\|g - Tf_1 - Tf_2 - \cdots - Tf_n\|_1 \leq \frac{1}{2^n}.$$

Letting $n \to \infty$ we have

$$g = \lim_{n\to\infty} T\left(\sum_{k=1}^{b} f_k\right) = Tf.$$

If any of the two vector spaces are not Banach, the theorem goes wrong.

Theorem 5.2.26

T is a bounded linear map if and only if the graph of T is closed.

Theorem 5.2.27: Bounded Inverse Theorem

Let $T : V \to W$ be a bounded linear map and bijective. Then T^{-1} is bounded and linear.

Theorem 5.2.28: Principle of Uniform Boundedness

Suppose V is Banach, W is a normed vector space, and $\mathcal{A}$ is a family of bounded linear maps from V to W such that

$$\sup\{\|Tf\|_1 : T \in \mathcal{A}\} < \infty, \forall f \in V,$$

which is somewhat "pointwise boundedness".
Then

$$\sup\{\|T\|_1 : T \in \mathcal{A}\} < \infty.$$

Proof for Theorem

$$V = \bigcup_{n=1}^{\infty}\{f \in V : \|Tf\|_1 \leq n, \forall T \in \mathcal{A}\} := \bigcup_{n=1}^{\infty} V_n$$

Since T is continuous, V_n is closed in V, By Baire's Theorem, there exists V_n that has nonempty interior:

$$\mathcal{B}(h, r) \subseteq V_n, \forall h \in V, r > 0.$$

Now suppose $g \in V$ and $\|g\|_1 < 1$. Thus,

$$rg + h \in \mathcal{B}(h, r) \subseteq V_n,$$

which implies that

$$\begin{aligned}
\|Tg\|_1 &= \|\frac{T(rg+h)}{r} - \frac{Th}{r}\|_1 \\
&\leq \frac{n}{r} + \frac{\|h\|_1}{r} \\
\implies \|T\|_1 &\leq \frac{n}{r} + \frac{\|Th\|_1}{r} \\
\implies \sup_{T\in\mathcal{A}} \{\|T\|_1\} &\leq \frac{n + \sup\{\|Th\|_1 : T \in \mathcal{A}\}}{r} < \infty
\end{aligned}$$

■